Frequency Analysis

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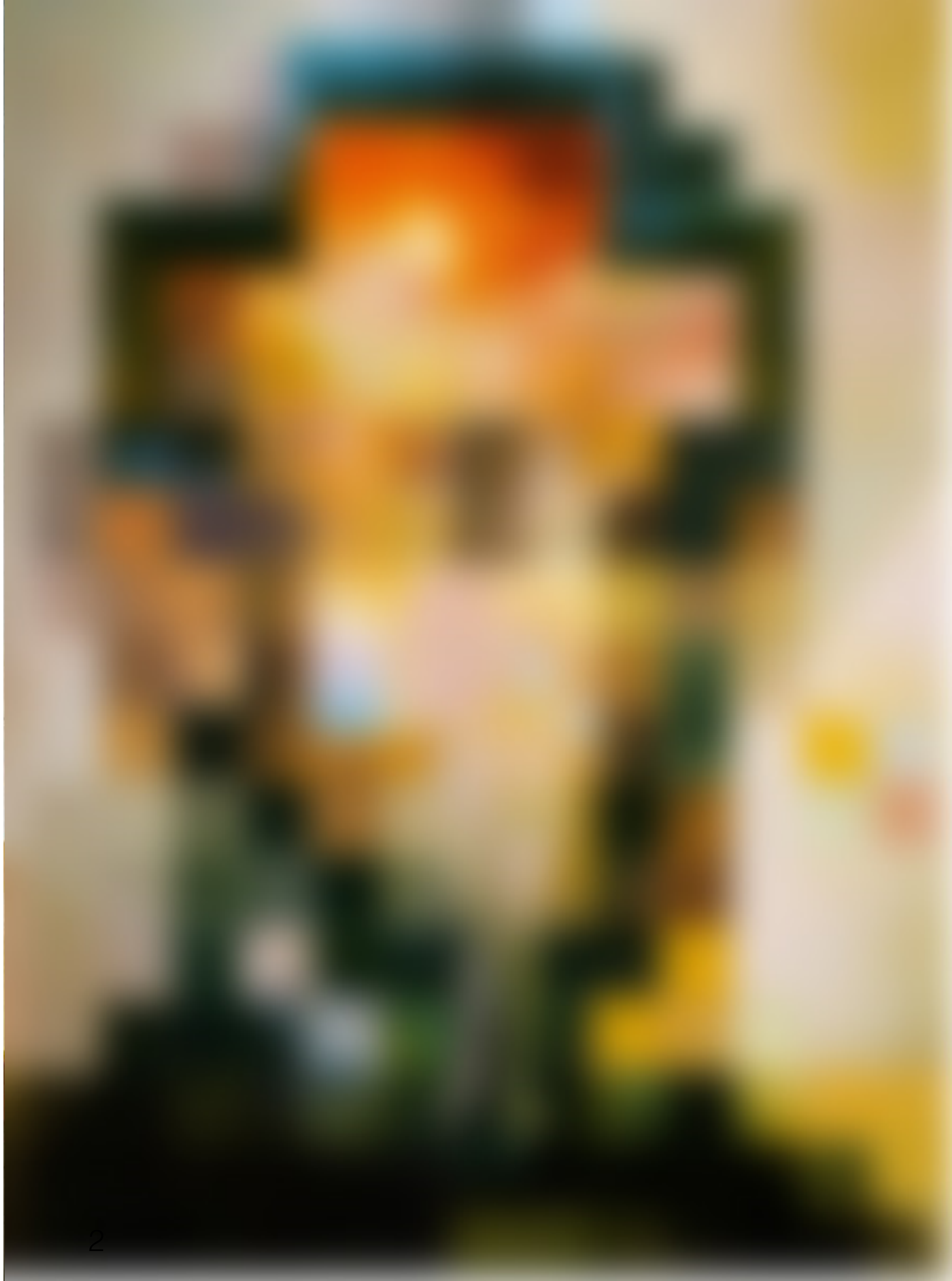
Salvador Dali

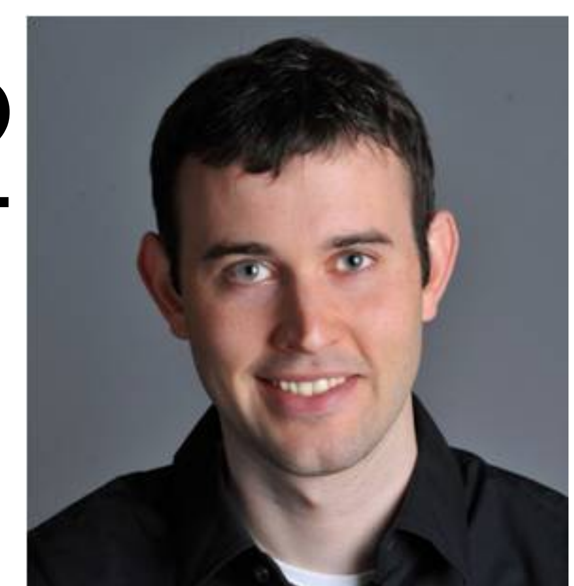
invented *Hybrid Images*



Why do we get two different distance dependent interpretations?

1

2



Why does lower resolution image still makes sense to us?

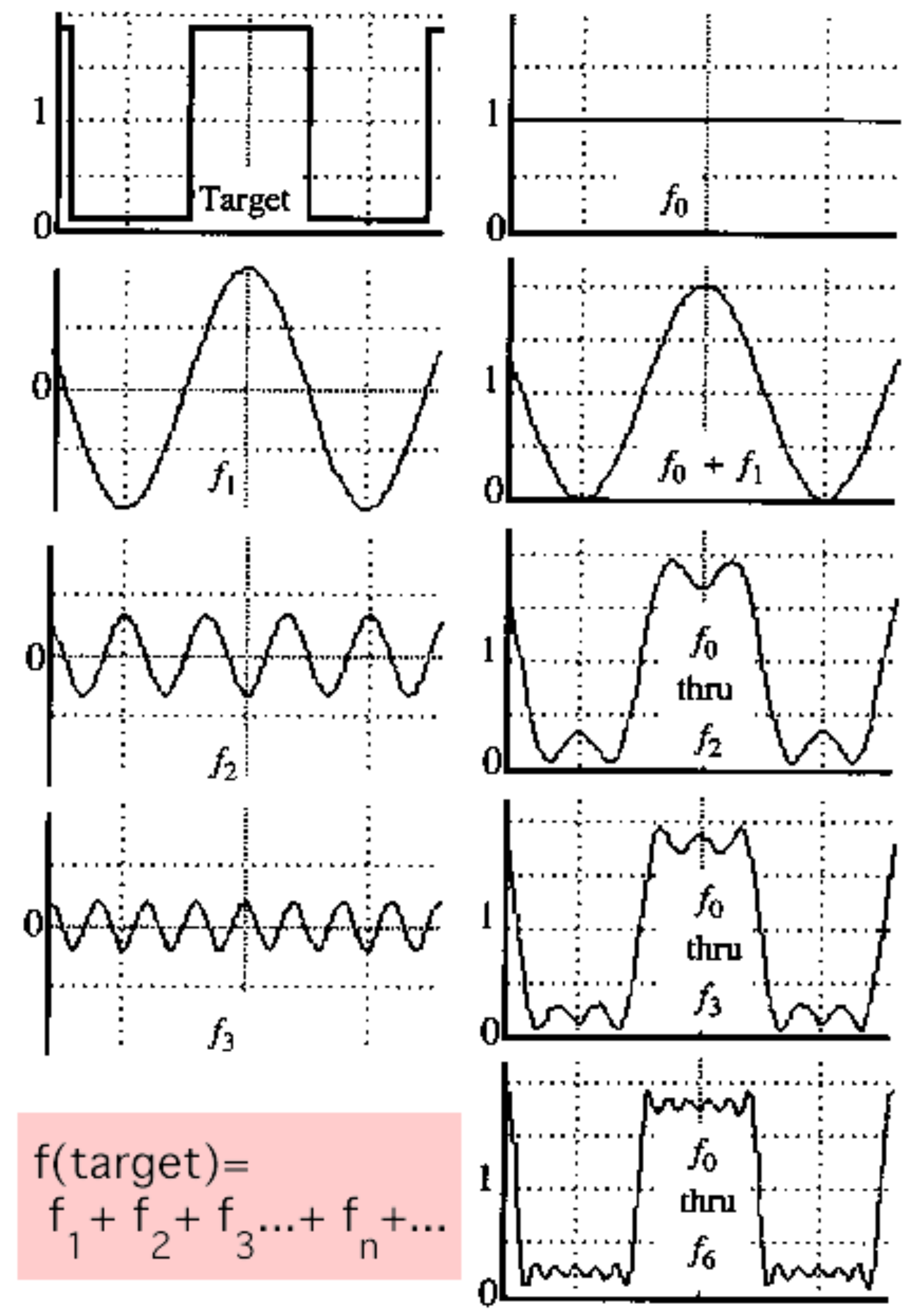
Jean Baptiste Joseph Fourier 1786 — 1830

- Any univariate function can be re-written as a sum of sines and cosines of different frequencies (1807)
  - No one believed him
  - Not translated into English until 1878
- It's true
  - Called *Fourier Series*

A sum of sines

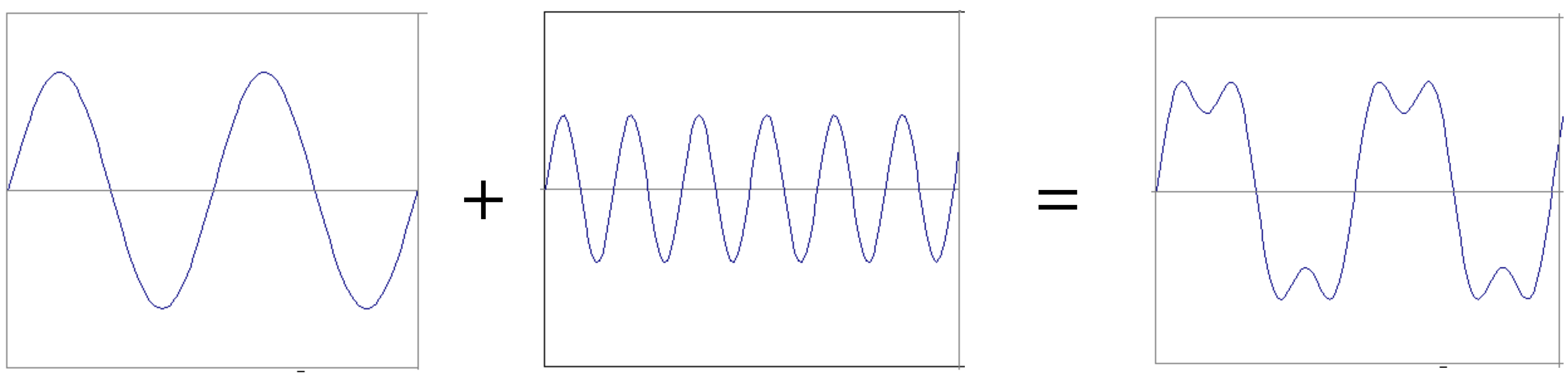
- Add enough of them to get any signal you want
- Basic building block

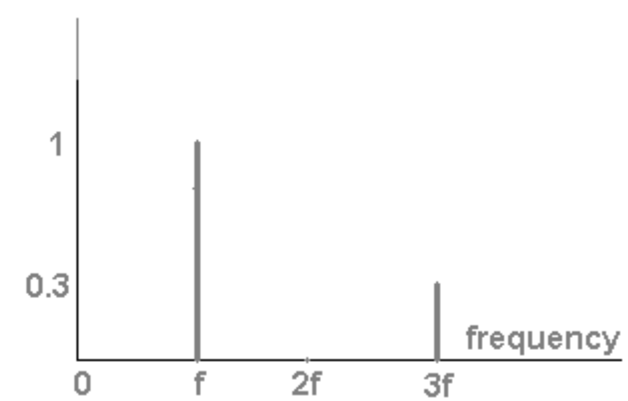
$$A \sin(wx + \phi)$$

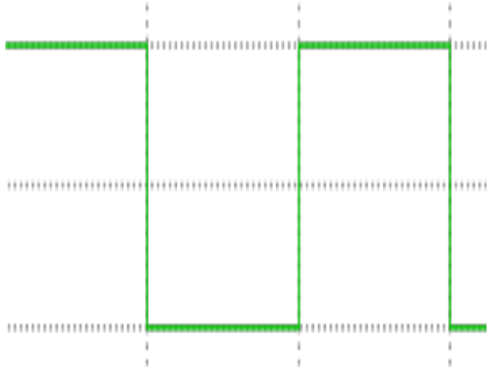


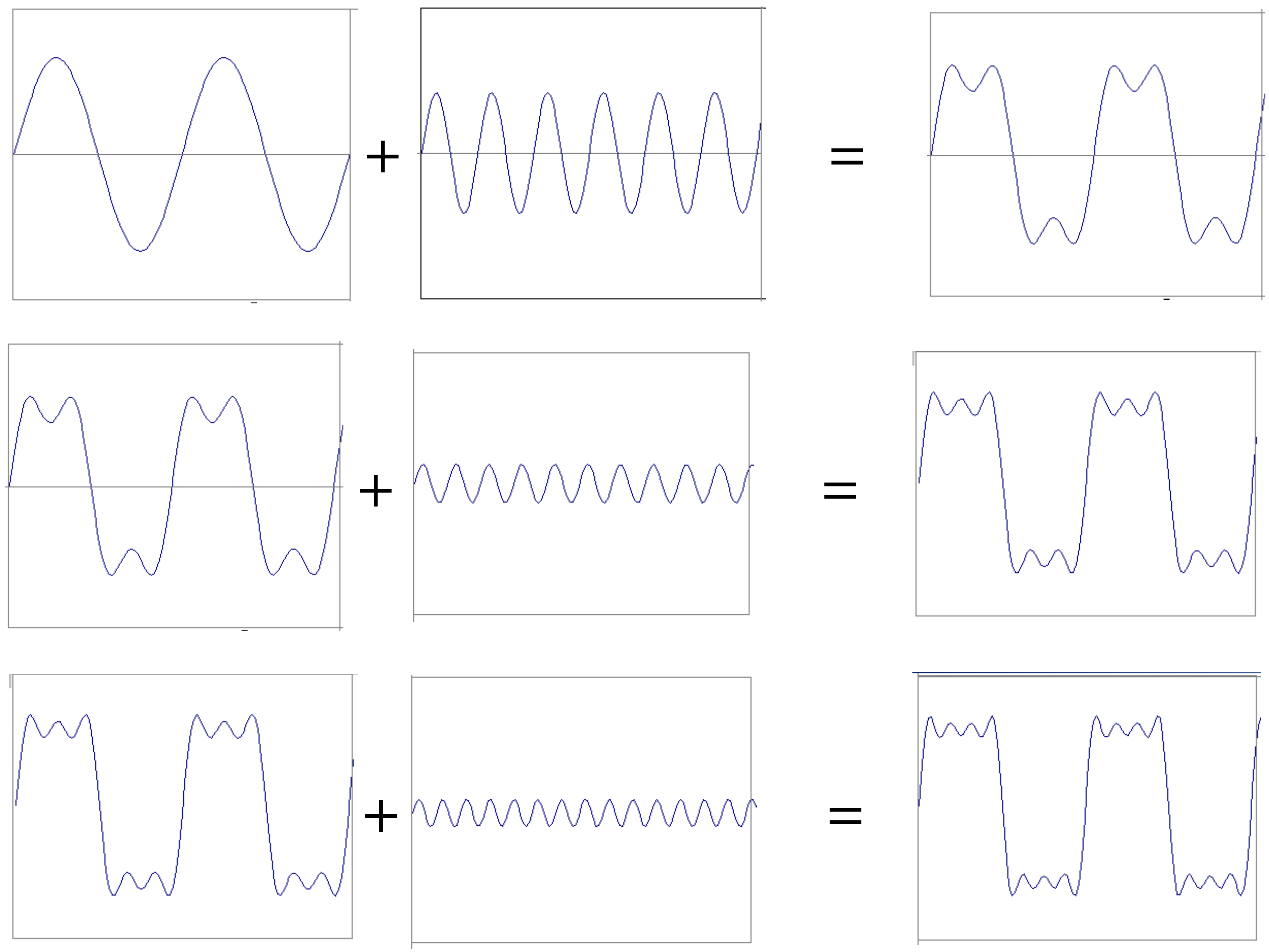
Frequency Spectra

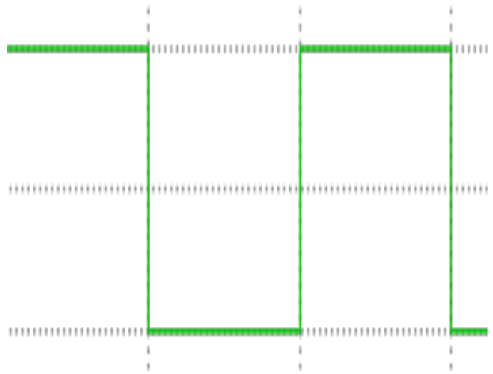
$$g(t) = \sin(2\pi f t) + \frac{1}{3}\sin(2\pi(3f)t)$$

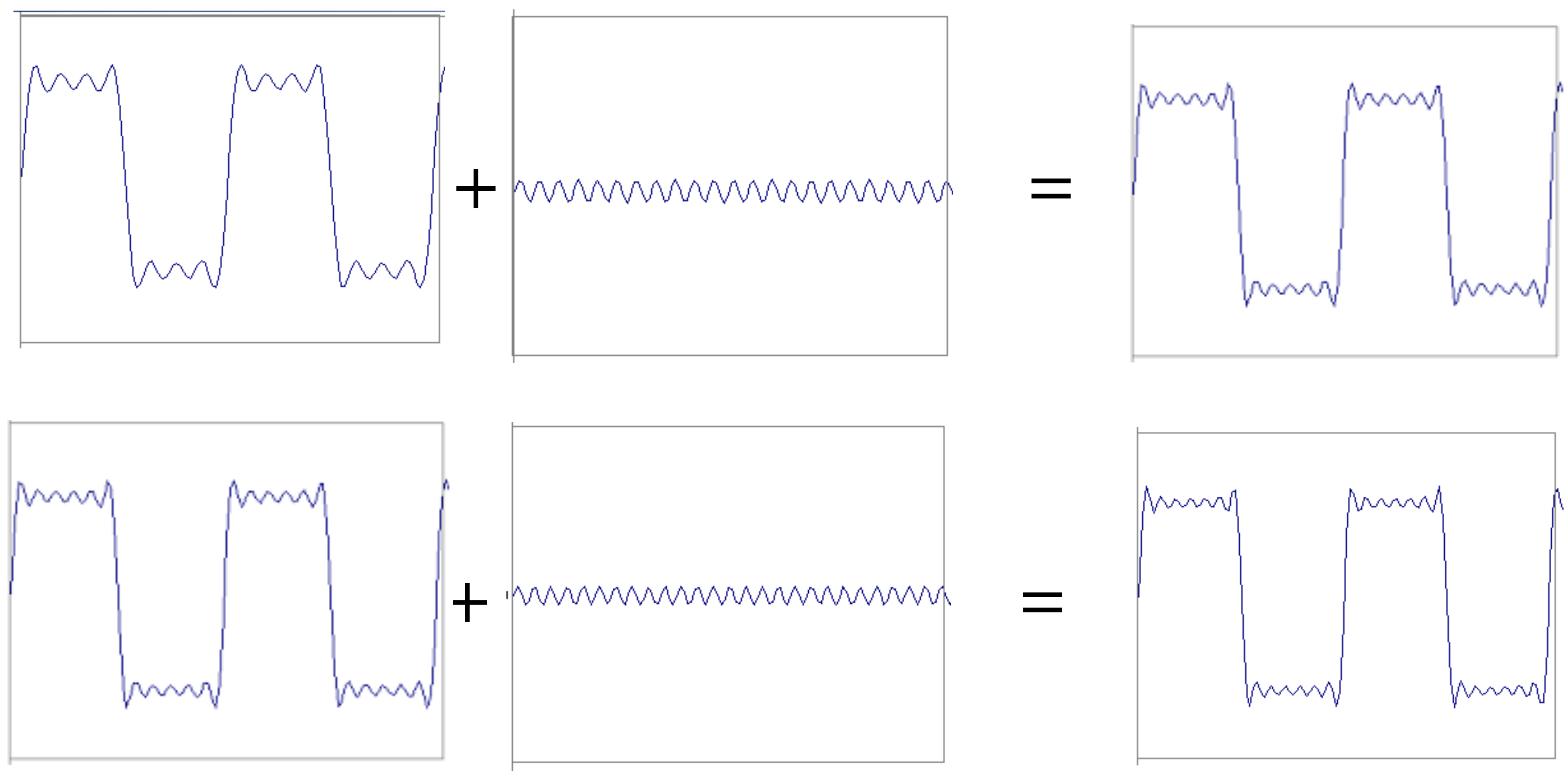


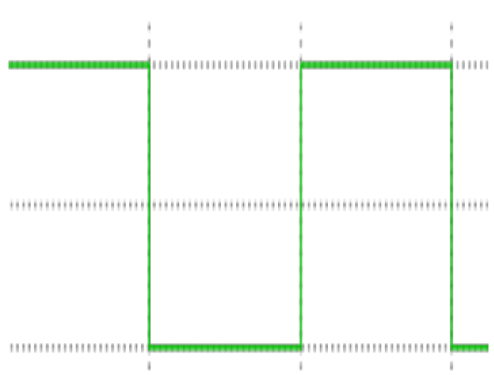




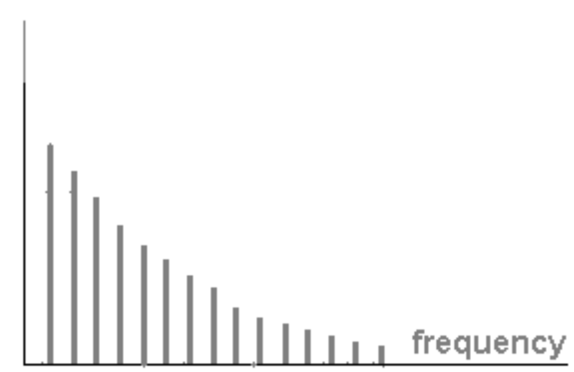






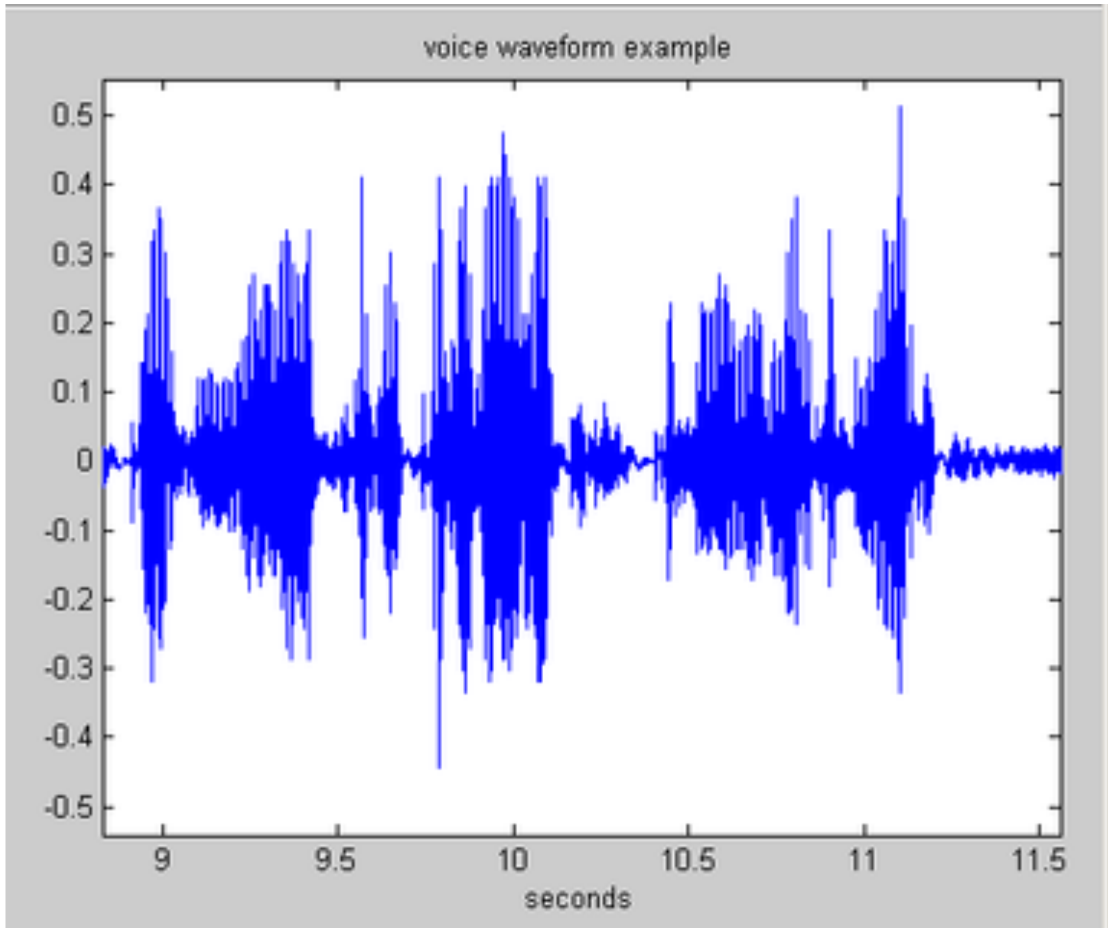


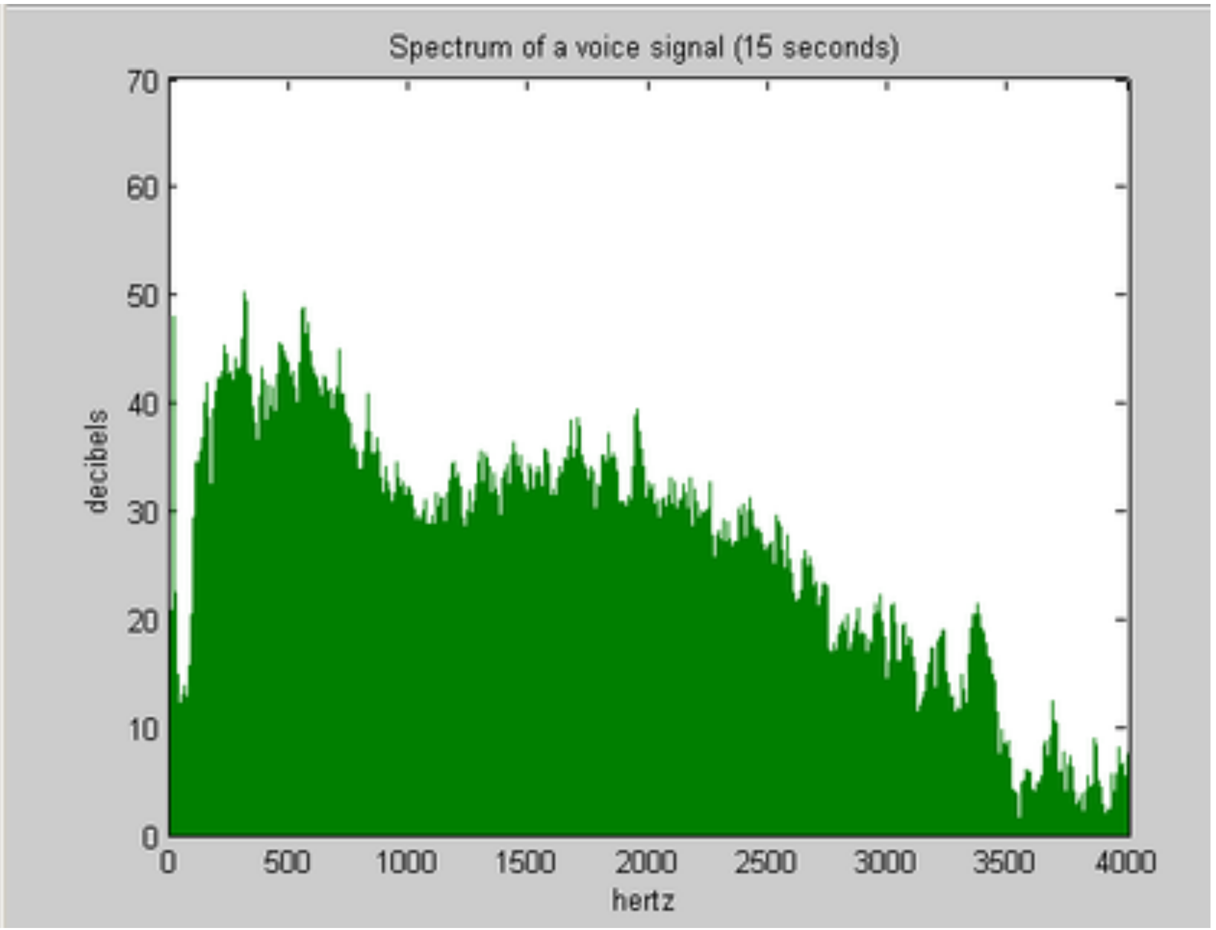
$$= A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi k t)$$



Example: Music

- We think of music in terms of frequencies at different magnitudes

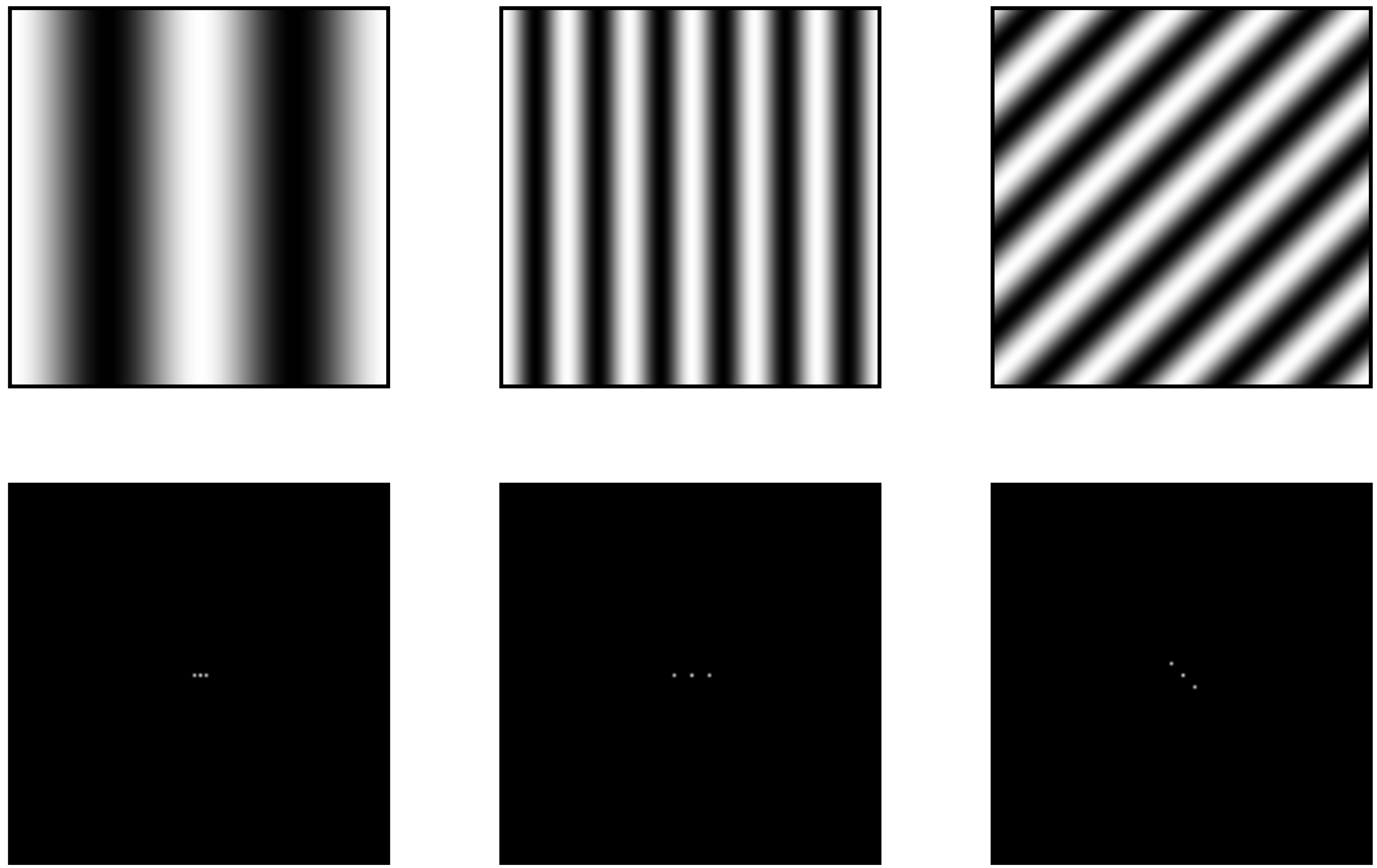




Fourier Analysis in Images

Intensity image

Fourier image



http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering

Fourier Transform

- Fourier transform stores the magnitude and phase at each frequency
  - Magnitude encodes how much signal is at a particular frequency

$$A = \pm\sqrt{R(w)^2 + I(w)^2}$$

  - Phase encodes spatial information (indirectly)

$$\phi = \arctan \frac{I(w)}{R(w)}$$

  - For mathematical convenience, these are often represented as real and complex numbers

Fourier Transform

- Fourier transform

$$\mathcal{F}\{f(x)\} = \hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

- Inverse Fourier transform

$$f(x) = \mathcal{F}^{-1}\{\hat{f}(w)\} := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$$

Discrete Fourier Transform

- A sequence of N complex numbers

$$x_0, x_1, x_2, \cdots, x_{N-1}$$

can be transformed into an N-periodic sequence of complex numbers

$$X_k := \sum_{n=0}^{N-1} x_n e^{-2\pi i k n / N}$$

- Fast Fourier Transform (FFT) is $N \log N$

Convolution Theorem

- Fourier transform of the convolution of two functions is the product of their Fourier transforms

$$\mathcal{F}\{g * h\} = \mathcal{F}\{g\}\mathcal{F}\{h\}$$

- The inverse Fourier transform of the product of two Fourier transform is the convolution of the two inverse Fourier transforms

$$\mathcal{F}^{-1}\{gh\} = \mathcal{F}^{-1}\{g\} * \mathcal{F}^{-1}\{h\}$$

Convolution in *spatial domain* is equivalent to multiplication in *frequency domain*

Properties of Fourier Transforms

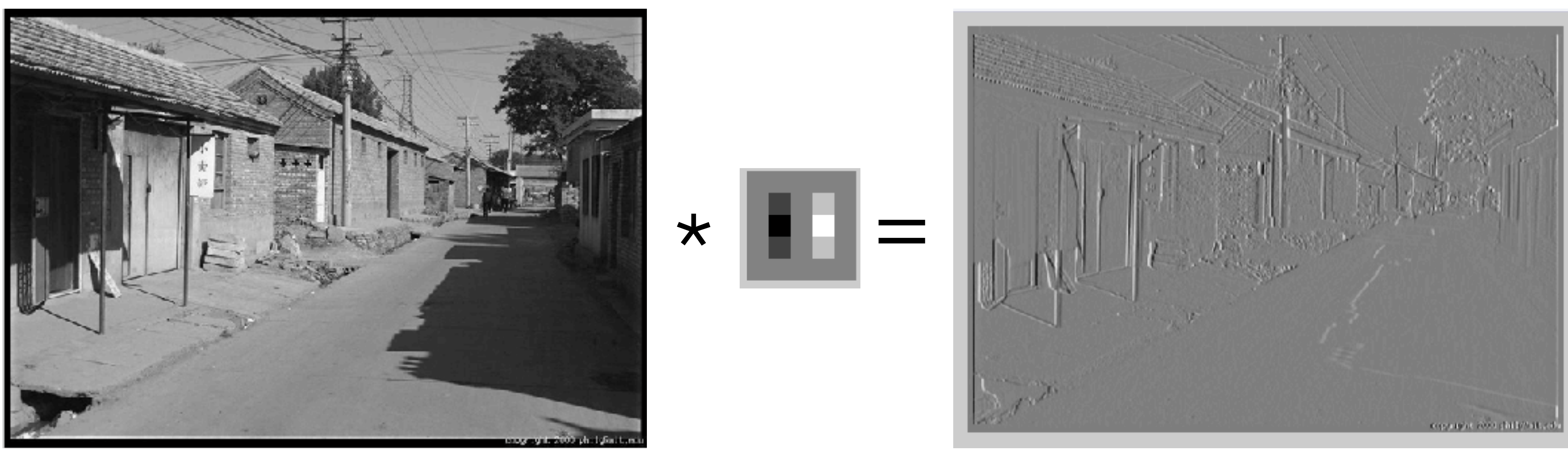
- Linearity

$$\mathcal{F}\{ag(x) + bh(x)\} = a\mathcal{F}\{g(x)\} + b\mathcal{F}\{h(x)\}$$

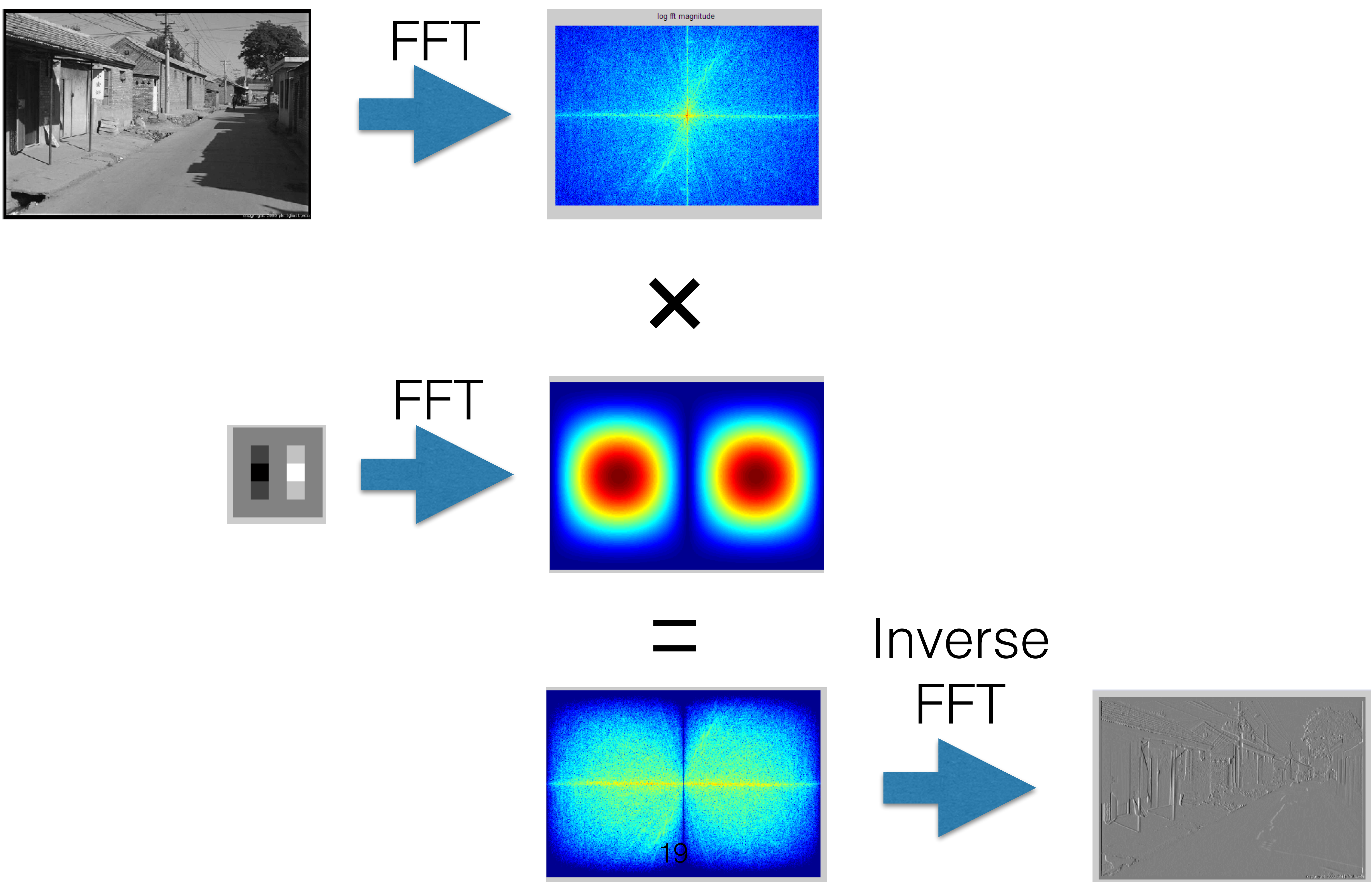
- Fourier transform of a real signal is symmetric around origin

- The energy of the signal is the same as the energy of its Fourier transform

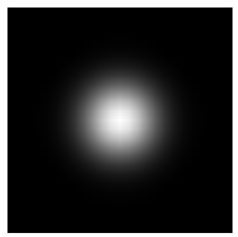
Filtering in Spatial Domain

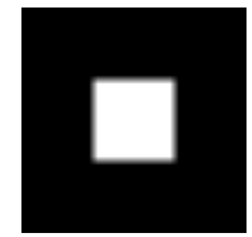


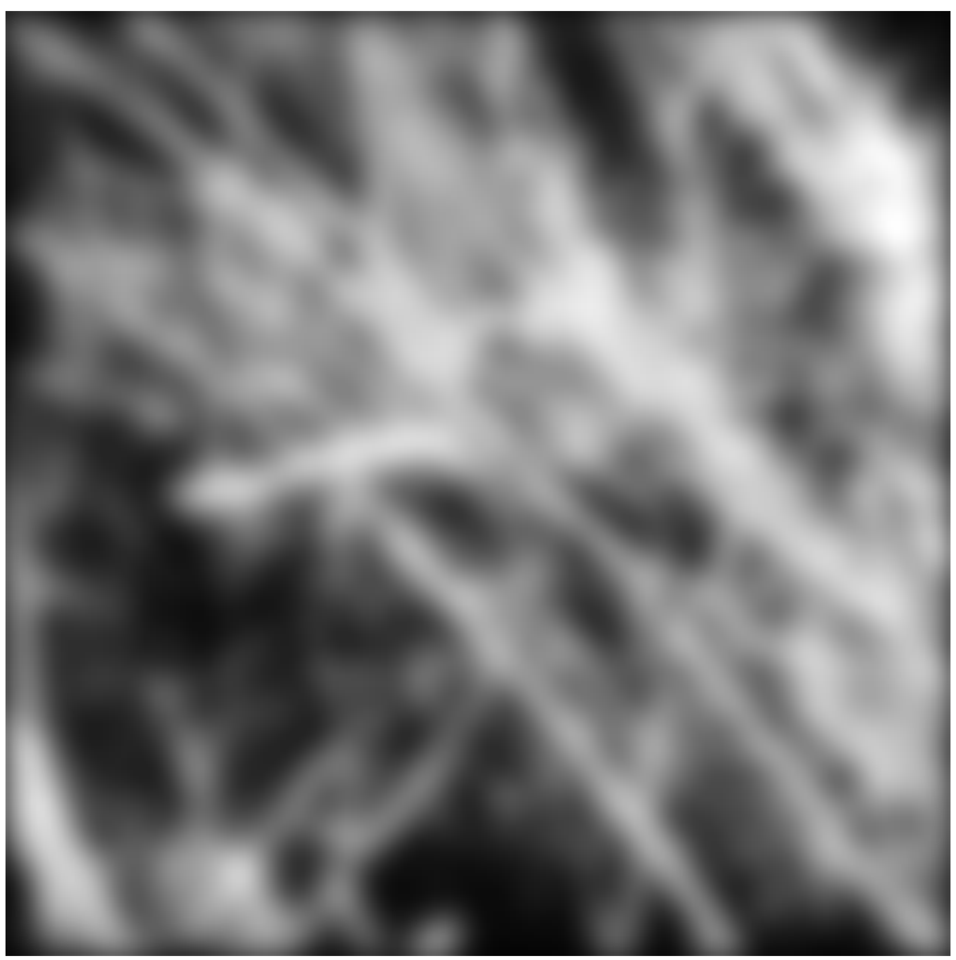
Filtering in Frequency Domain

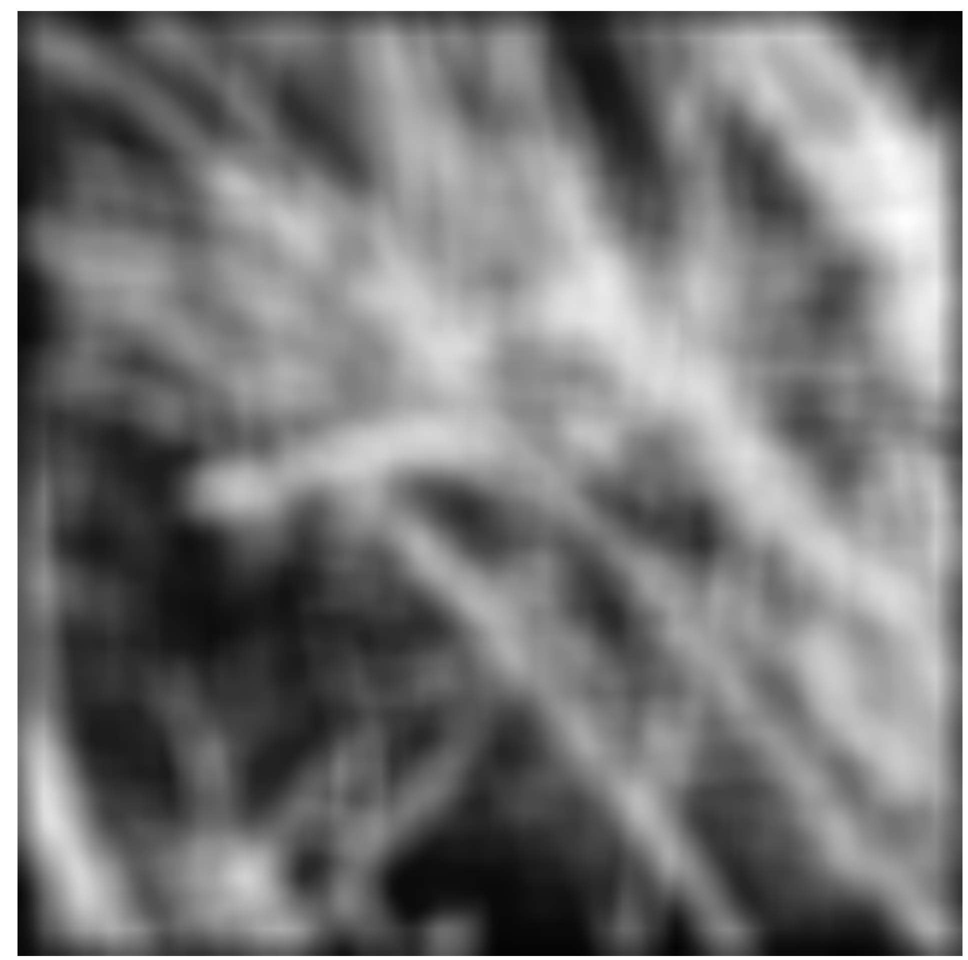


Why does Gaussian filter gives a smooth image, where as the square filter gives edgy artifacts?

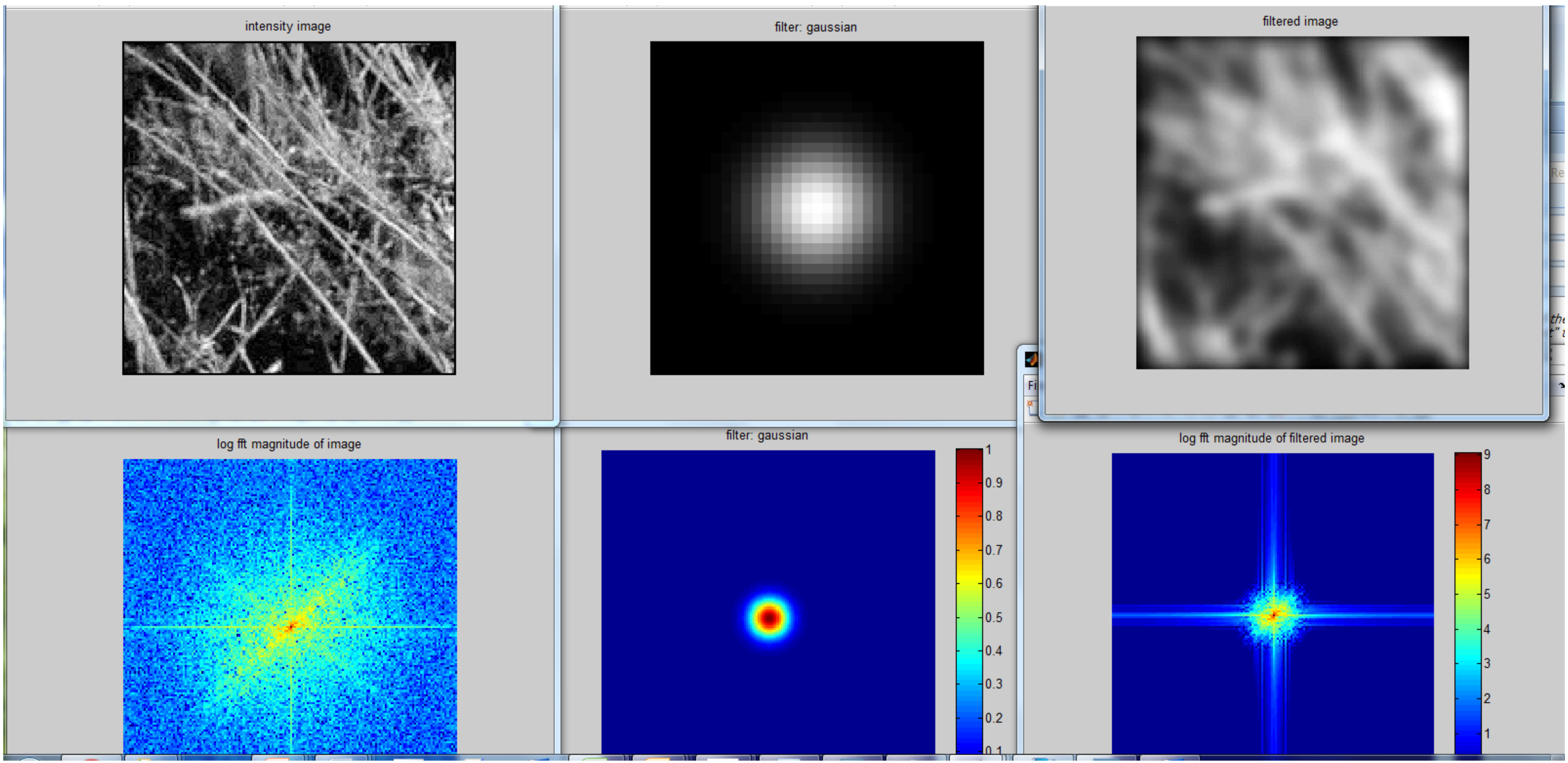
Gaussian 

Box filter 

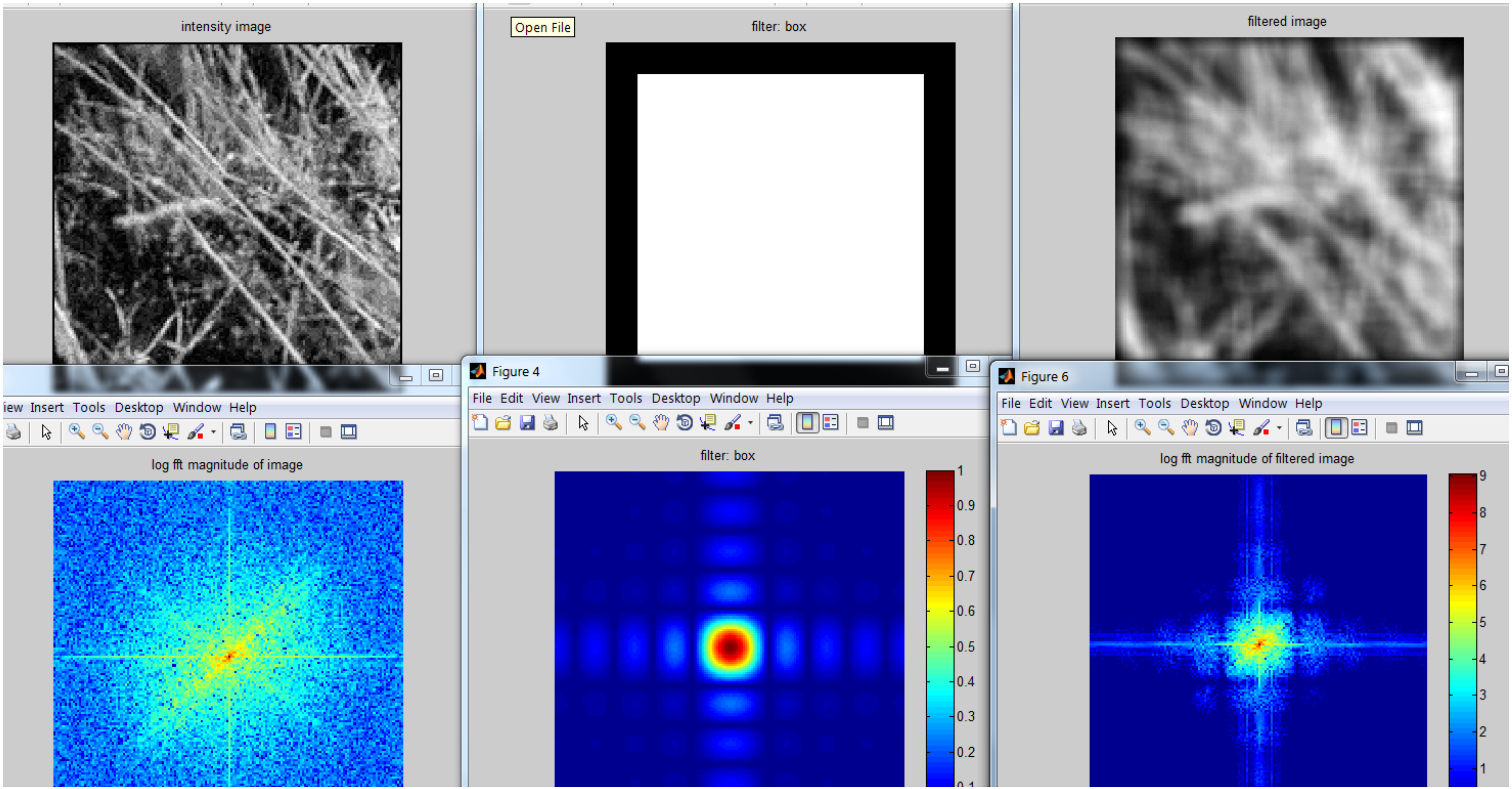




Gaussian


intensity image
filter: gaussian
filtered image
log fft magnitude of image
filter: gaussian
log fft magnitude of filtered image

Box Filter


intensity image
Open File
filter: box
filtered image
Figure 4
Figure 6
iew Insert Tools Desktop Window Help
File Edit View Insert Tools Desktop Window Help
File Edit View Insert Tools Desktop Window Help
log fft magnitude of image
filter: box
log fft magnitude of filtered image
1
0.9
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0.4
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4
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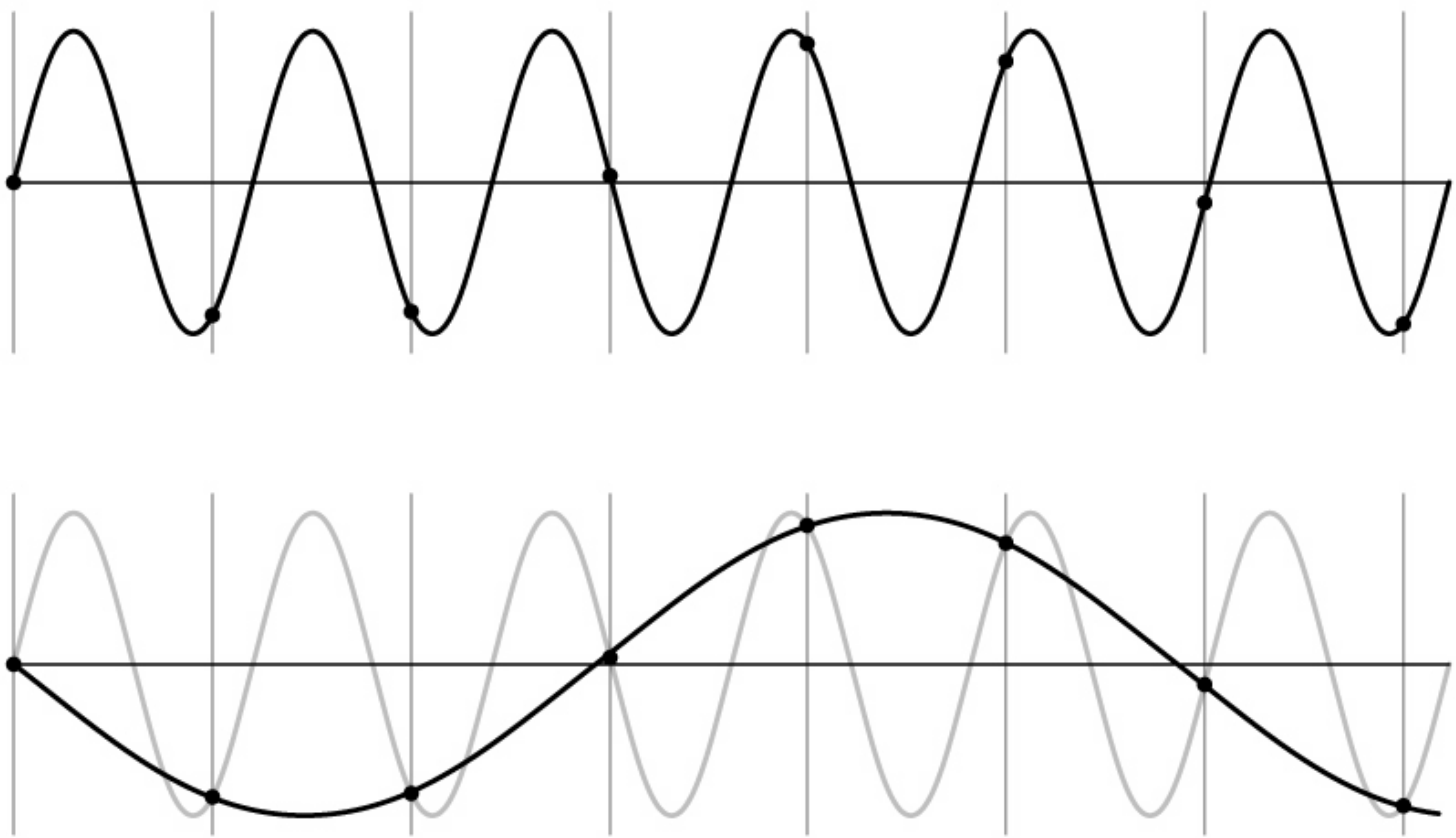
2
1

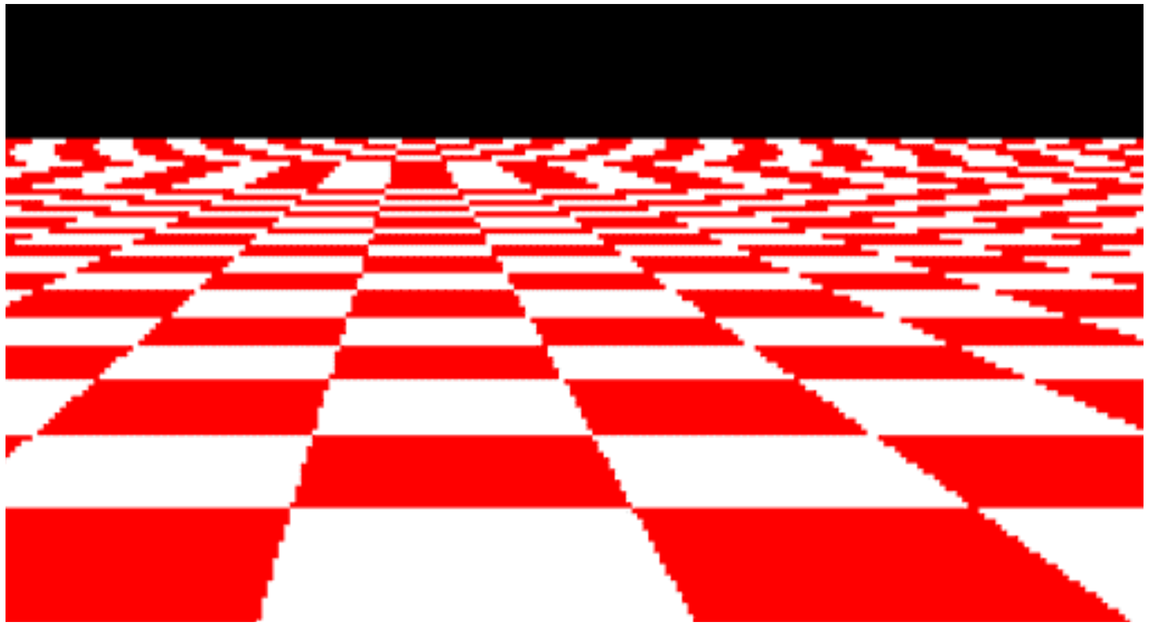
Why does lower resolution image still makes sense to us?

Aliasing Problem

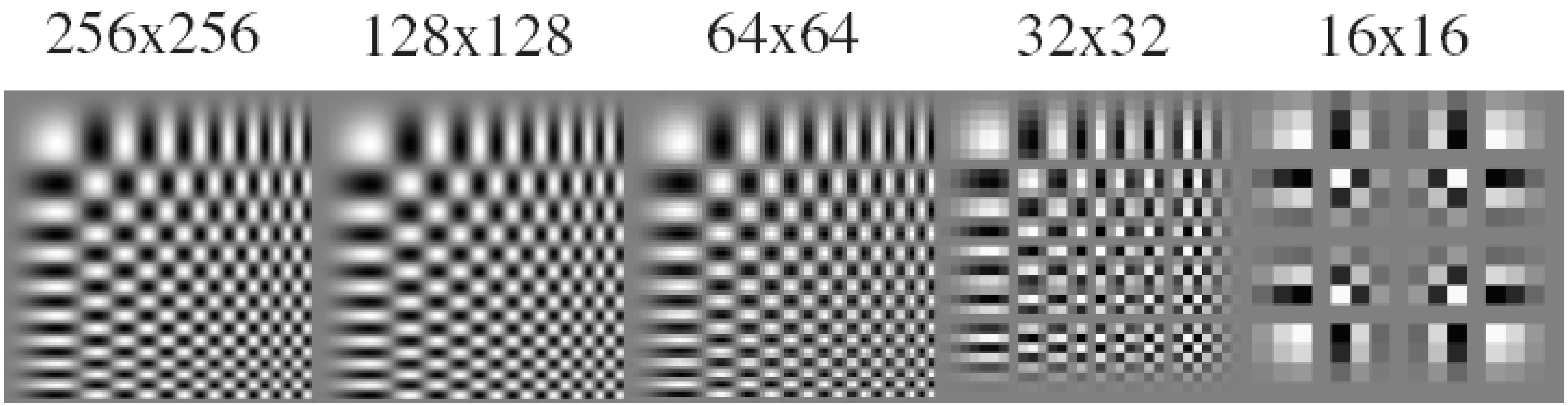


Aliasing Problem

- Sub-sampling can result in aliasing artifacts
  - Wagon wheels rolling the wrong way in movies
  - Checkerboards disintegrate in ray tracing
  - Striped shirts look funny



Sampling and Aliasing



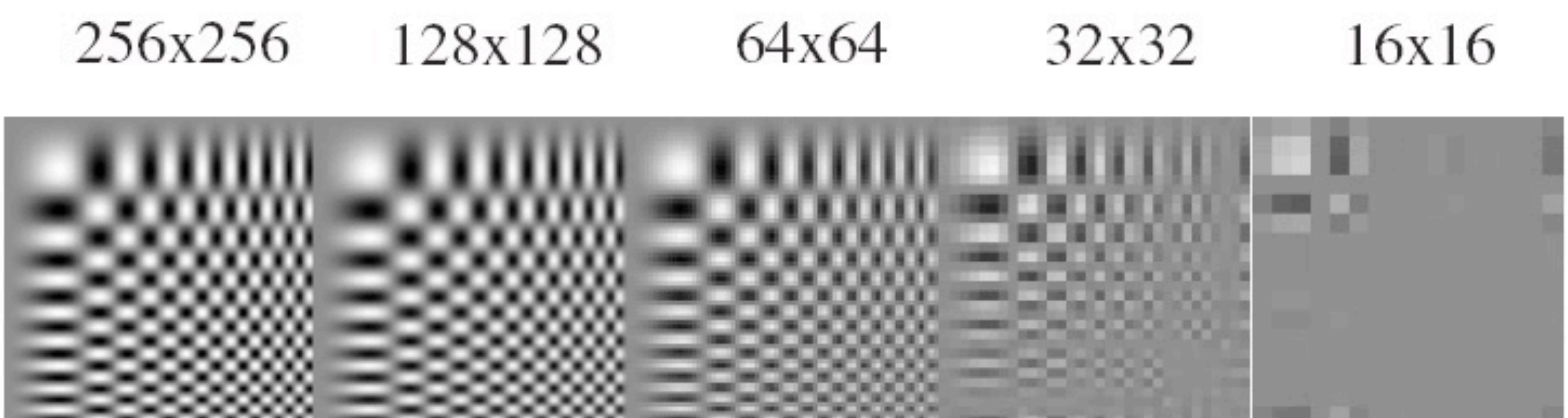
Nyquist-Shannon Sampling Theorem

- When sampling a signal at discrete intervals, the sampling frequency must be twice the maximum frequency of the signal

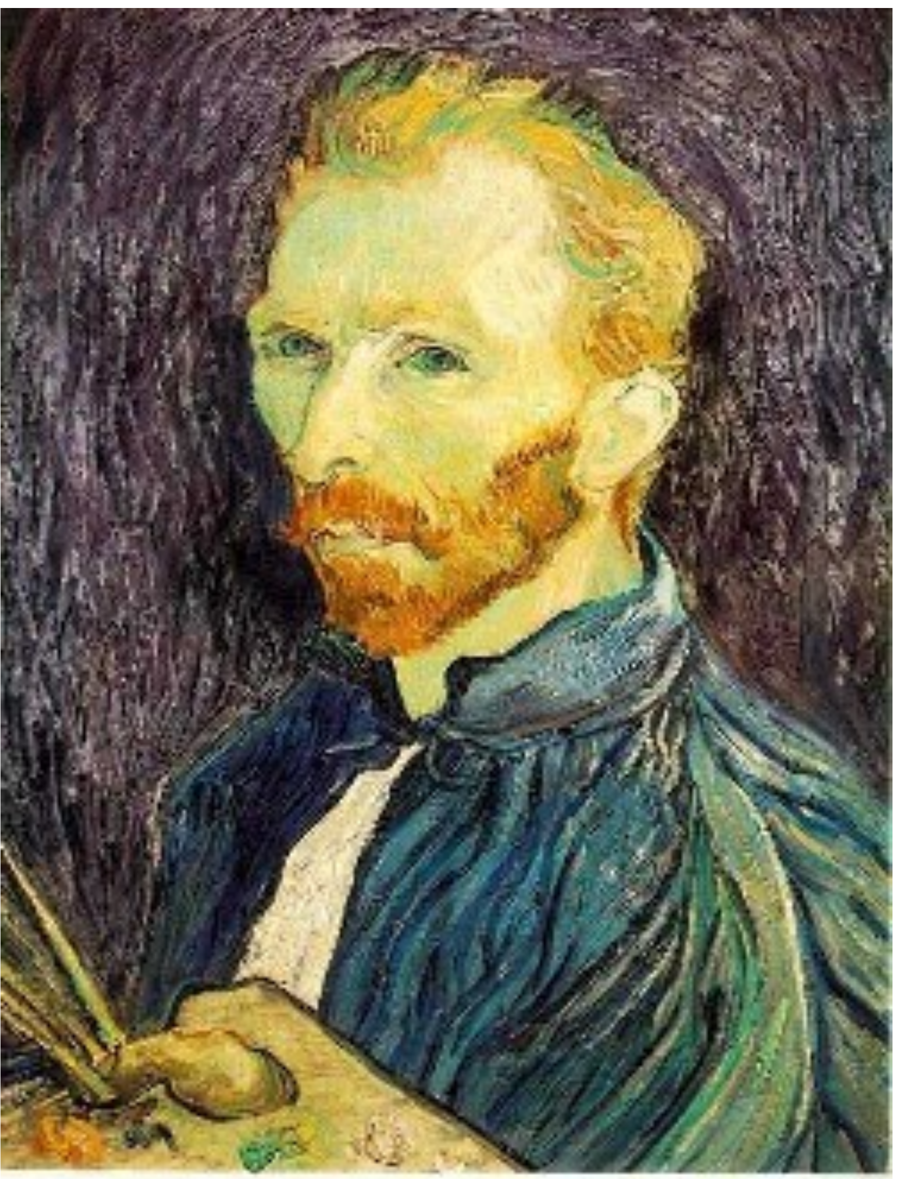
- This will allow us to perfectly reconstruct the original signal from its samples

Anti-Aliasing

- Sample more often
- Get rid of all frequency that are greater than half the sampling frequency
  - We will loose information
  - But still better than aliasing artifacts



Submsampling without Pre-Filtering

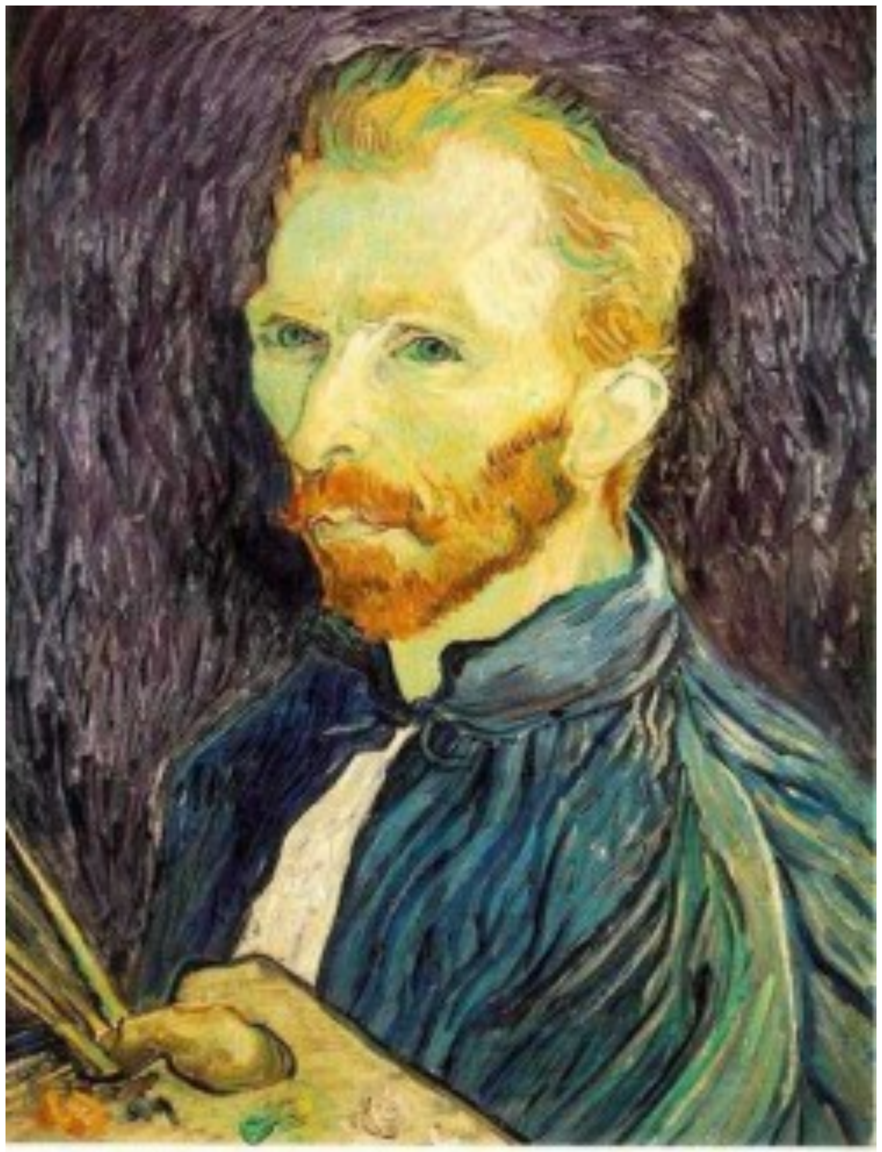


1/2

1/4
2x zoom

1/8
4x zoom

Subsampling with Gaussian Pre-Filtering

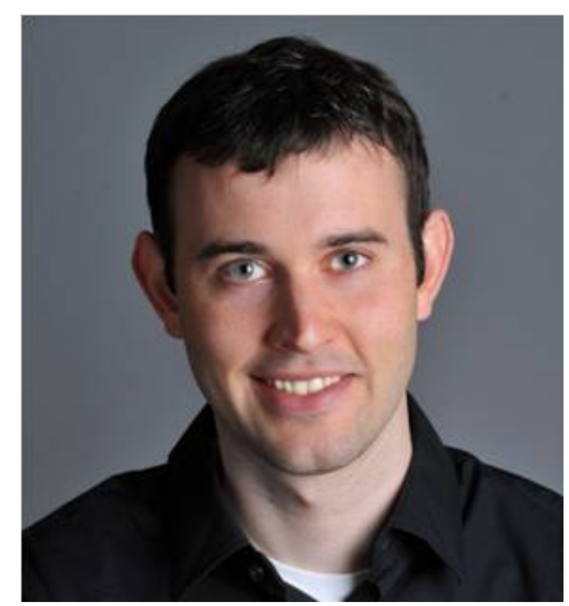


1/2

1/4
2x zoom

1/8
4x zoom

Why do we get two different distance dependent interpretations?



Summary

- Image filtering in frequency domain
  - Fourier analysis
- Image filtering in frequency domain is vs for auto correlation
- Images are smooth — image compression
- Low-pass filter before smapling