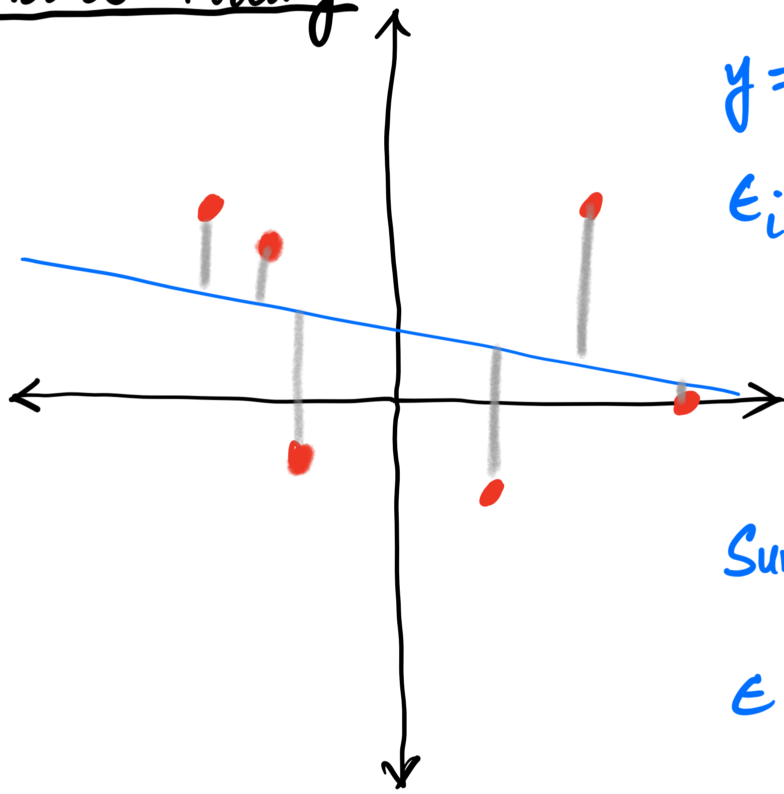
Model Fitting



$y = mx + c$

$\epsilon_i = y_i - mx_i - c$

Sum of squared errors

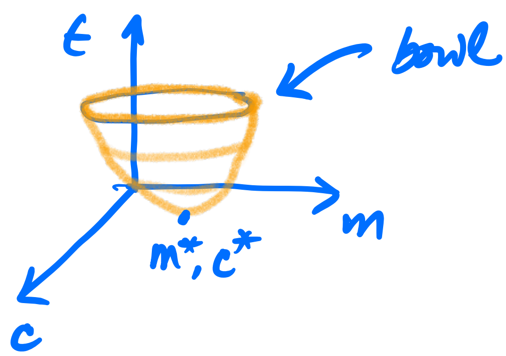
$$\epsilon = \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

? ?

In order to find the best-fitting line we find m and c by solving the following optimization problem:

$$m^*, c^* = \underset{m,c}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

$\because$ The error ϵ is quadratic we can solve the optimization by setting $\frac{\partial \epsilon}{\partial c} = 0$ and $\frac{\partial \epsilon}{\partial m} = 0$.



$$\epsilon = \sum_{i=1}^{n} (y_i - m x_i - c)^2$$

$y_2 - m x_2 - c$

$$= \left\| \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} \right\|^2$$

$$= \left\| \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \right\|^2$$

$$= \| \vec{y} - A\vec{p} \|^2$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$= \vec{y}^T \vec{y} - 2(A\vec{p})^T y + (A\vec{p})^T A\vec{p}$$

Compute $\dfrac{\partial \epsilon}{\partial \vec{p}} = ?$

$$\frac{\partial \epsilon}{\partial \vec{p}} = -2A^T\vec{y} + 2A^T A\vec{p}$$

Set $\dfrac{\partial \epsilon}{\partial \vec{p}} = 0$ and solve for $\vec{p}$.

$$-2A^T\vec{y} + 2A^TA\vec{p} = 0$$

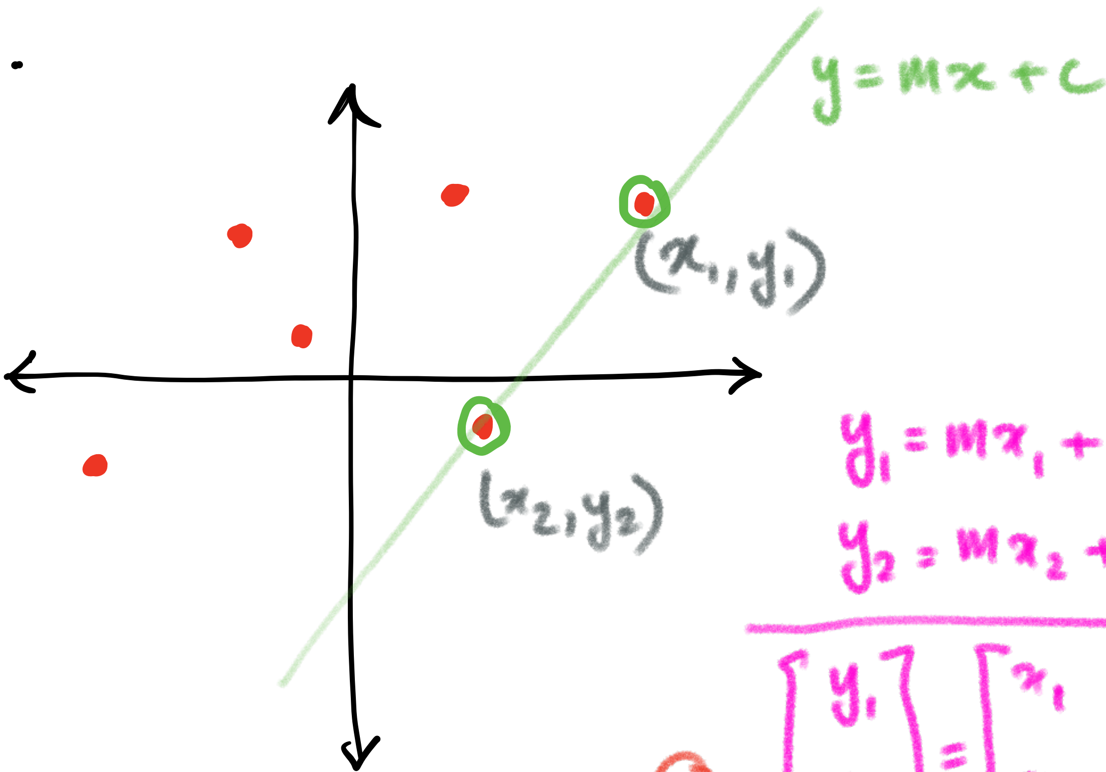
$$\Rightarrow \quad A^TA\vec{p} = A^T\vec{y}$$

$$\Rightarrow \quad \underbrace{\vec{p}}_{\begin{bmatrix} m \\ c \end{bmatrix}} = \underbrace{\underbrace{(\underbrace{A^T}_{2\times n}\underbrace{A}_{n\times 2})^{-1}}_{2\times 2}\underbrace{A^T}_{2\times n}\underbrace{\vec{y}}_{n\times 1}}_{2\times 1}$$

least squares fit of a <u>linear model</u>

in our case the lowly 2D line.

Detour.



$$y_1 = mx_1 + c$$
$$y_2 = mx_2 + c$$

(1) $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix}\begin{bmatrix} m \\ c \end{bmatrix}$$

* CHANGE OF NOTATION.

① $A\vec{x} = \vec{b}$

BACK TO OUR NOTATION.

$$A\vec{p} = \vec{y}$$

How to solve this equation?

$$\vec{p} = A^{-1}\vec{y}$$

We want to fit the line to multiple points.

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix}$$

is not a square matrix.

$$\Rightarrow \boxed{A}\,\vec{p} = \vec{y}$$

$$\Rightarrow A^T A \vec{p} = A^T \vec{y}$$

linear regression.

$$\Rightarrow \vec{p} = \underbrace{(A^T A)^{-1} A^T}_{\text{pseudo-inverse}} \vec{y}.$$

Detour.

Line: $y = mx + c$

linear in a, b, c

Quadratic model: $y = ax^2 + bx + c$ $\quad \boxed{y = f(x; a,b,c)}$

①

$\epsilon_i = y_i - a x_i^2 - b x_i - c \qquad (x_1, y_1) \cdots (x_n, y_n)$

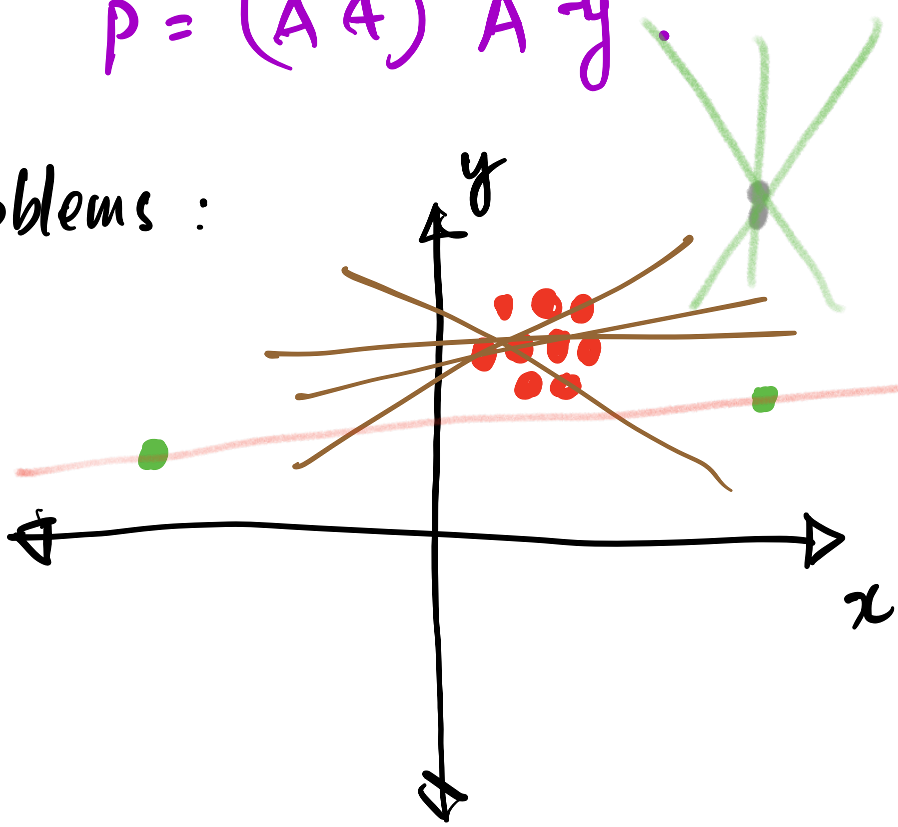
✗

② or ③

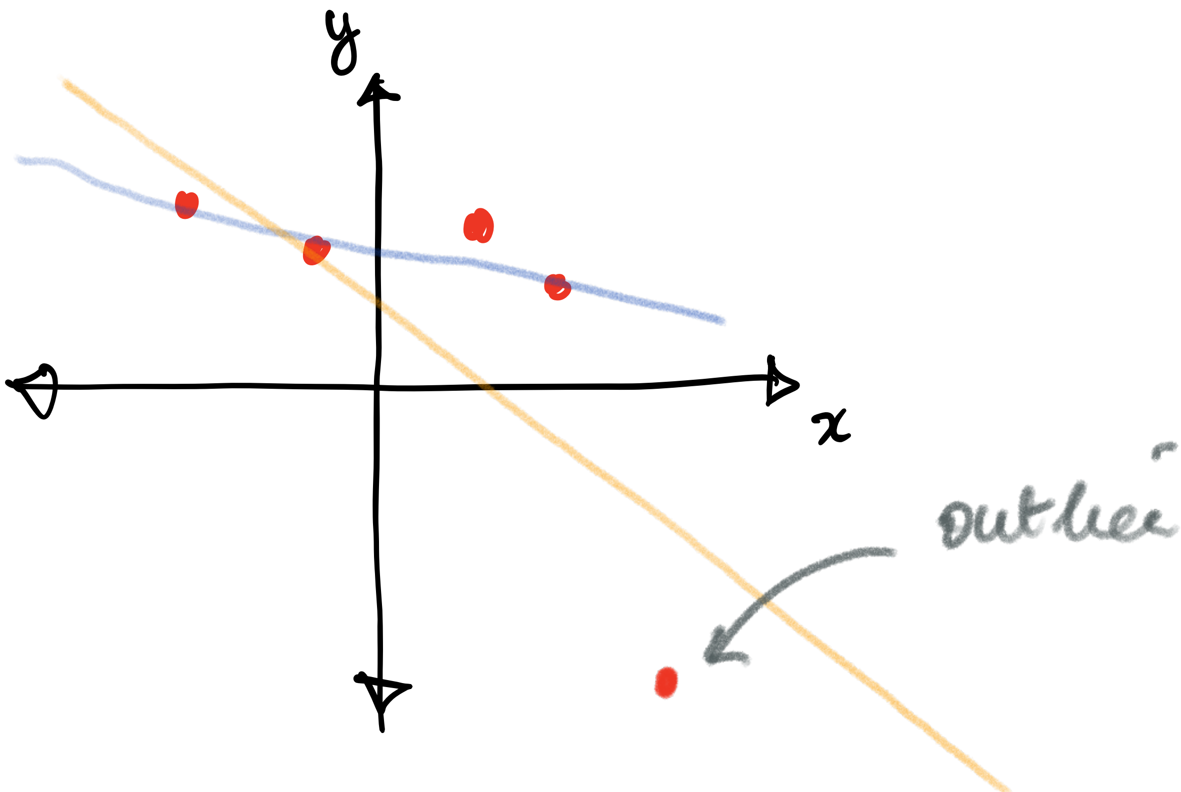
$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad A = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ & \vdots & \\ x_n^2 & x_n & 1 \end{bmatrix} \qquad \vec{p} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow \quad \vec{p} = (A^T A)^{-1} A^T \vec{y}.$$

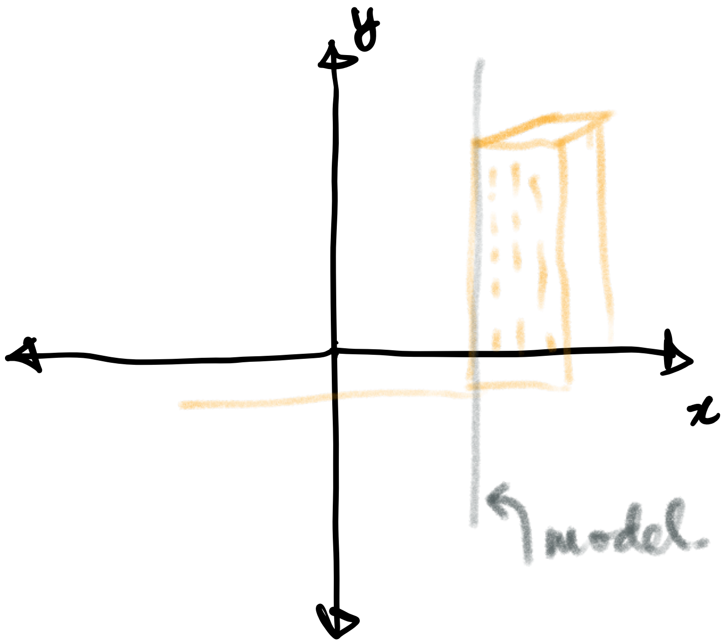
Problems :



ill-conditioned data.



CHALLENGE



$y = mx + c$

$\downarrow$

Fitting using edge pixel locations.

$(x_1, y_1) \cdots (x_n, y_n)$

Oops. Can't fit. line is vertical.

Slope $m \to \infty$.

Use Implicit Equation of a Line:

$ax + by + c = 0$