Model Fitting (Linear Regression)

(i) Finding objects/elements in a image
(ii) Image and scene understanding

* 2D line fitting (least squares)

line: $y = \overbrace{mx + c}^{\text{model}}$

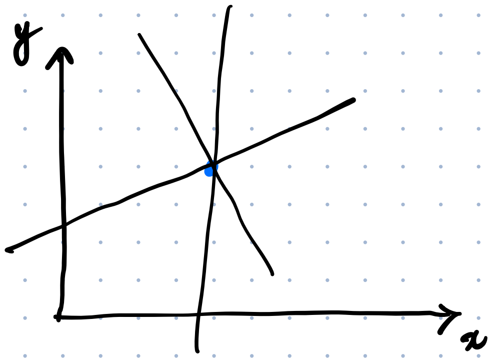
slope

y-intercept

m, c are model parameters.

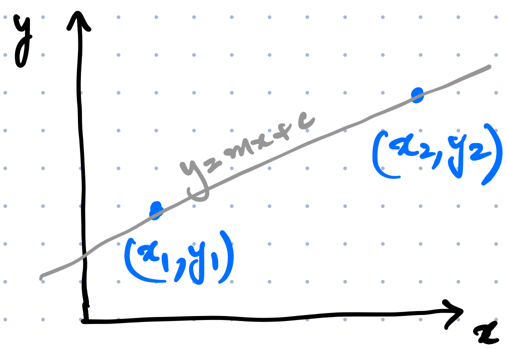
The problem of line fitting is the problem of estimating model parameters m & c given data.

(a)



not enough data to estimate m and c.

(b)



exactly the right amount of data to fit a line.

→ two unknown values (m, c)

two points give rise to two equations and we can solve these equations to estimate m and c.

$$\begin{aligned} y_1 &= \boxed{m}x_1 + \boxed{c} \\ y_2 &= \boxed{m}x_2 + \boxed{c} \end{aligned} \longrightarrow \overset{\vec{b}\ (\vec{y})}{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}} \overset{\vec{x}\ (\vec{p})}{\begin{bmatrix} c \\ m \end{bmatrix}}$$

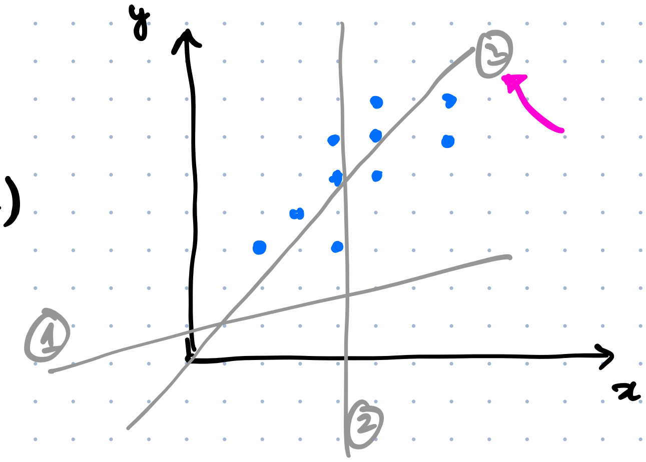
?

$A\vec{x} = \vec{b}$ system of linear equations.

Solution $\vec{x} = A^{-1}\vec{b}$ ⟶ $\vec{p} = A^{-1}\vec{y}$

$$\Rightarrow \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(c)



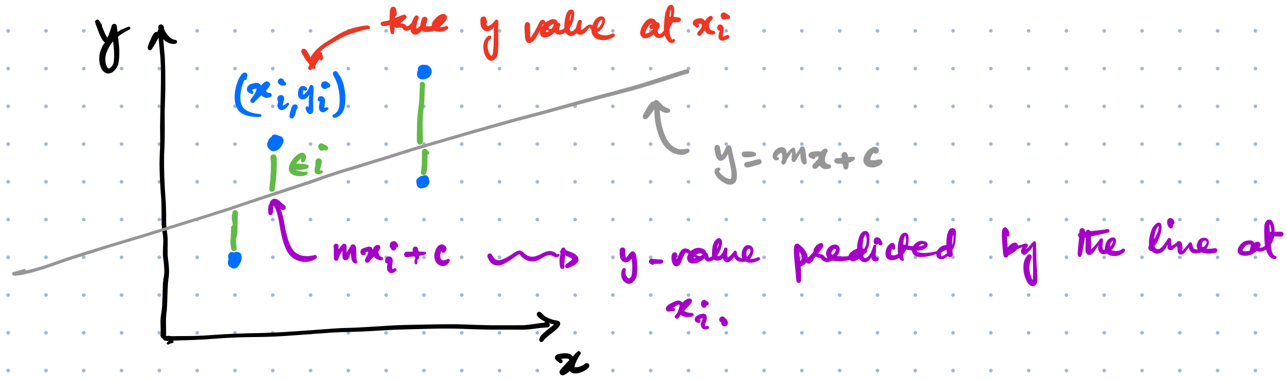
(i) no single line can pass through all these points.

(ii) we need a line that best represents this data.

line ③ $\boxed{\text{best}}$ represents this data.

* How do we define "best"?

Find the line that minimizes the least square error.



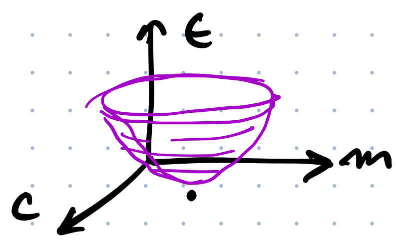
$$\epsilon_i = \left(y_i - (mx_i + c)\right)^2$$

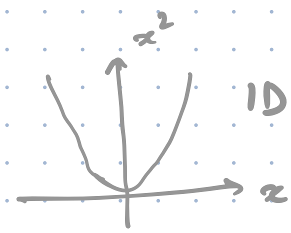
Total error:

$$\epsilon = \sum_{i=1}^{N} \left(y_i - \underline{\underline{m}}x_i - \underline{\underline{c}}\right)^2$$

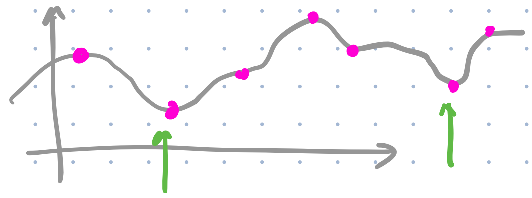
To fit the model ($y = mx + c$) we are asked to minimize ϵ.

Observation: this ϵ is a fn. of two variables m and c.





For ϵ at its smallest value, its derivative is ϕ.



This allows us to estimate m and c (i.e., fit the line) by computing $\frac{\partial \epsilon}{\partial m}$ and $\frac{\partial \epsilon}{\partial c}$ and setting them equal to 0.

$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

$\|\vec{v}\|^2 = a^2 + b^2$

* Re-writing ϵ

$$\epsilon = \sum_{i=1}^{n} (y_i - m x_i - c)^2$$

$$= \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} \right\|^2$$

$$= \| \vec{y} - A \underset{\substack{\uparrow \\ \text{parameters}}}{\vec{p}} \|^2$$

To solve it

$$\frac{\partial \epsilon}{\partial \vec{p}} = 0$$

$$\Rightarrow \vec{p} = \underbrace{(A^T A)^{-1} A^T}_{\text{pseudo-inverse of } A} y$$

Recall that in case (b):

$$\vec{y} = A \vec{p}$$

in this case (c), A is not square.

$$\Rightarrow \quad A^T \vec{y} = A^T A \vec{p}$$

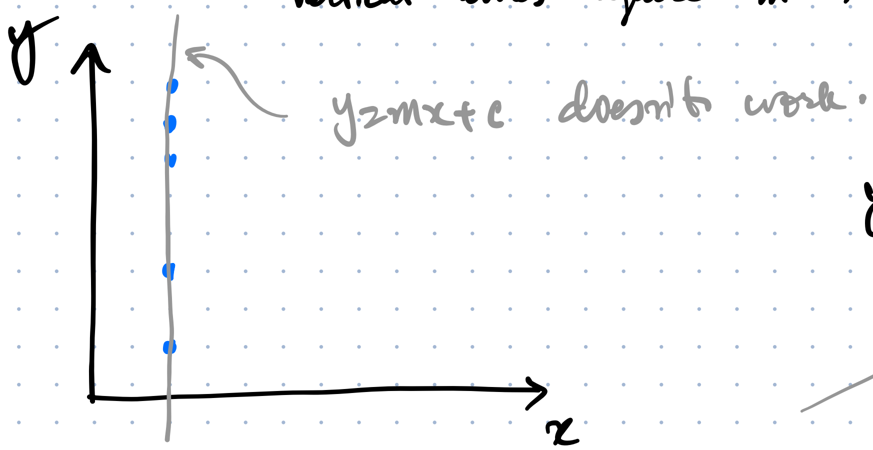
$$\Rightarrow \quad A^T A \vec{p} = A^T \vec{y}$$

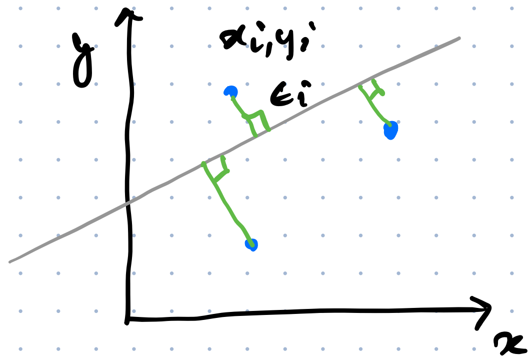
$$\Rightarrow \quad \vec{p} = (A^T A)^{-1} A^T \vec{y} .$$

* It is very beneficial to transform a computer vision problem to $\boxed{A\vec{p} = \vec{y}}$ form.

* Observation: $y = mx + c$ can not represent vertical lines.

$\uparrow$ vertical lines require $m \to \infty$





$$ax + by + c = \phi$$

$\uparrow \quad \uparrow \quad \uparrow$

model parameters

gives rise to system of linear equations of the form $\boxed{A\vec{p} = 0}$?

(step 1) $\quad c = -a\langle x\rangle - b\langle y\rangle$

$\langle x\rangle$: average x value (mean of x)

$\langle y\rangle$: average y value (mean of y)

$$\epsilon = \left\| \underbrace{\begin{bmatrix} x_1 - \langle x\rangle & y_1 - \langle y\rangle \\ \vdots & \\ x_n - \langle x\rangle & y_n - \langle y\rangle \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}} \right\|^2$$

$$= \| A\vec{p} \|^2$$

Solution:

$$p^* = \arg\min_{p} \|Ap\|^2 \quad \text{s.t.} \quad p^T p = 1$$

once we have a, b, we can compute c.

The solution is the smallest eigenvectors of A^TA.

* Theme

(i) $A\vec{p} = \vec{y}$

(ii) $A\vec{p} = \phi$

(iii) Think really hard and see if you are able to express a computer vision problem into either $A\vec{p} = \vec{y}$ or $A\vec{p} = 0$.

Aside: $y = mx + c$