February 6, 2026

* For any fn. of period T

$$f(t) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n t/T}.$$

$$\hat{f}(n) = \frac{1}{T} \int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\, dt.$$

(i) Phase $\phi_n = \arctan\left(\frac{b_n}{a_n}\right)$

(ii) Amplitude $A_n = \sqrt{a_n^2 + b_n^2}$

(iii) Frequency $f_n = n/T$

* Discrete Fourier Transform

A sequence of numbers: $x_0, x_1, \ldots, x_{N-1}$

$$X_k = \sum_{n=0}^{N-1} x_n e^{2\pi i \frac{k}{N} n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

Recover the original

$$x_n = \frac{1}{N} \sum_{n=0}^{N-1} X_k e^{2\pi i \frac{k}{N} n}$$

Complexity $O(N^2)$

* FFT (fast Fourier transform)

$O(N \log N)$

$N = 1\,000\,000$ $\quad N^2 = 1\,000\,000\,000\,000$ (1 trillion)

$N \log N = 20\,000\,000$ (20 million)

* Notation

$$\hat{f}(n) = \mathcal{F}\{f(t)\}$$

Fourier transform

signal in freq.

Fourier operator

signal in time

$f(t) = \mathcal{F}^{-1}\{\hat{f}(n)\}$ Inverse Fourier transform.

* Properties of Fourier transform.

(i) Linearity: $a\,x(t) + b\,y(t) \longleftrightarrow a\,X(f) + b\,Y(f)$

signal in time $\qquad F(x(t))$

(ii) Time scaling: $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$

(iii) Time shifting: $x(t-t_0)$

does not change freq., phase changes

(iv) Modulation: multiply a signal by a sinusoid in time, the signal shifts to an entirely new spectrum in frequency.

(v) Convolution:

$$x(t) * y(t) \longleftrightarrow X(f)\cdot Y(f)$$

(vi) Parseval's theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Example: Convolve a 1920×1920 image with a 65×65 kernel.

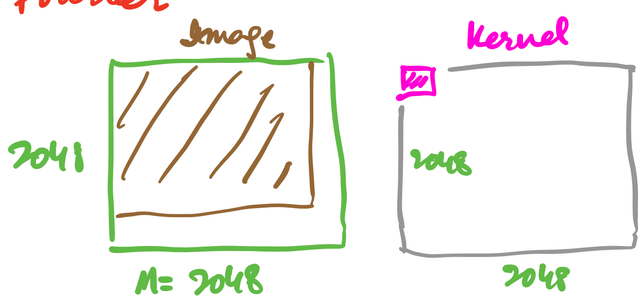
(1) Spatial

Operations per pixel $\approx 65 \times 65 \times 2$

total pixels $= 1920 \times 1920$

total operations ≈ 31.2 billion operations

(2) Fourier



pixels : M^2 where $M = 2048$

$M^2 = (2048)^2 = 4,194,304$

FFT operation: $M^2 \log M$ = 461,373,440 (image)

461,373,440 (kernel)

4,194,304 (multiplications)

461,373,440

~ 1.4 billion operations