

Polynomial Approximation

Computational Photography (CSCI 3240U)

Faisal Z. Qureshi

<http://vclab.science.ontariotechu.ca>



Today's lecture

- How to compute image derivatives by fitting polynomials to 1D image patches?
 - Taylor series expansion around a patch center
 - Least square fitting of a system of linear equations

Image as a surface in 3D

Consider a gray-scale image $I(x, y)$ then the height of the surface at (x, y) is $I(x, y)$. The surface passes through the 3D point $(x, y, I(x, y))$.

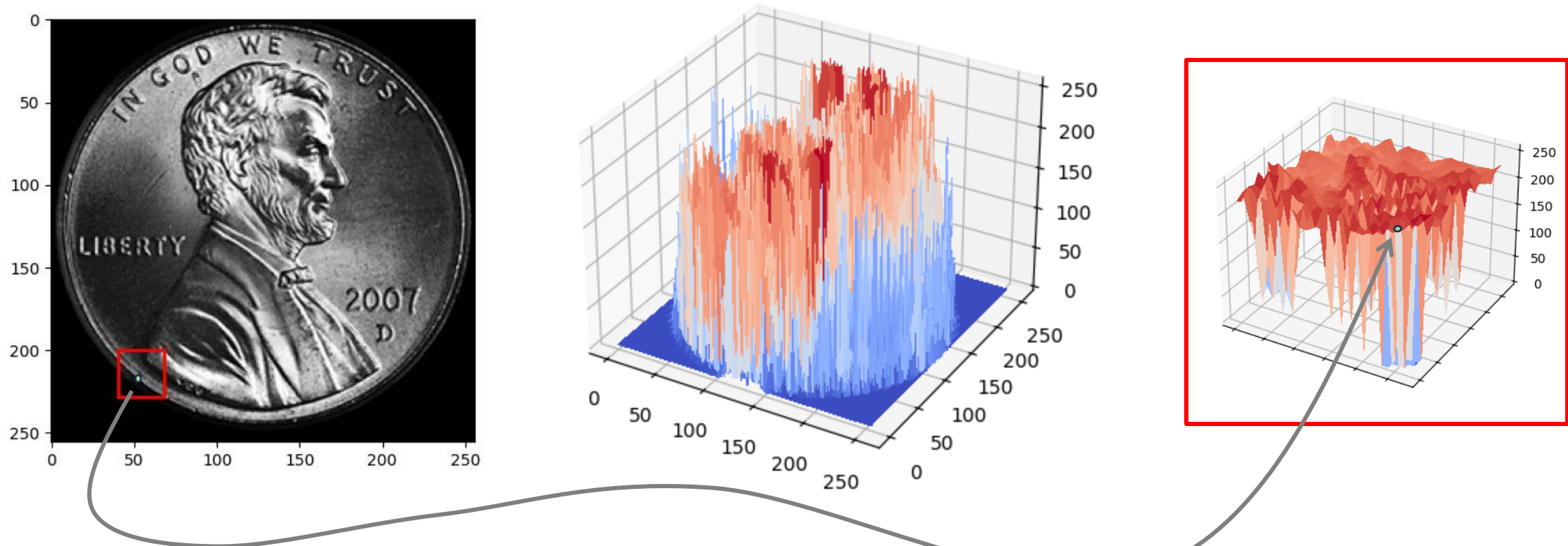
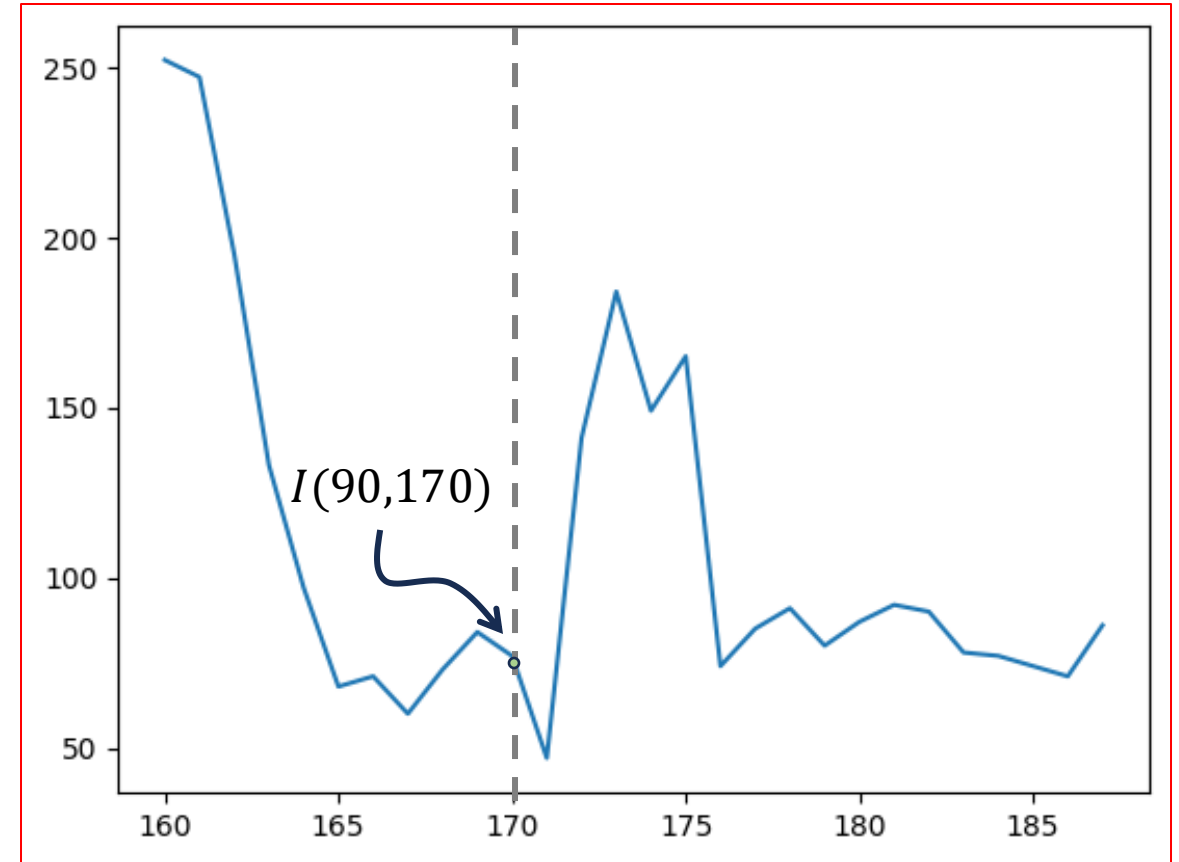
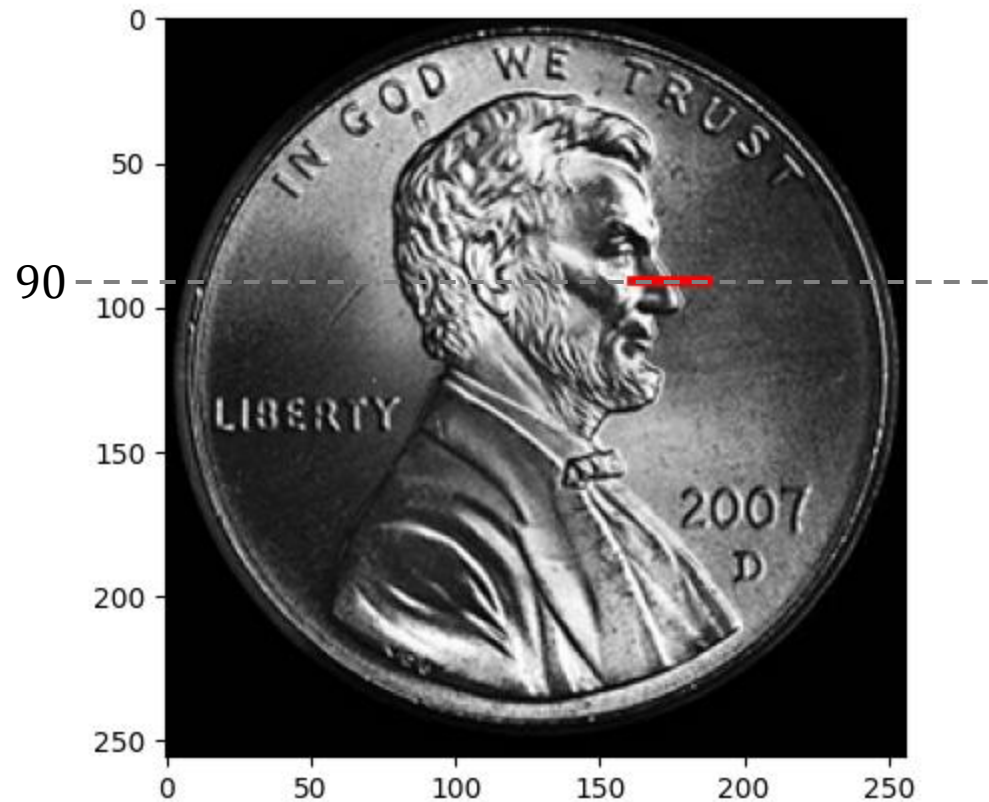
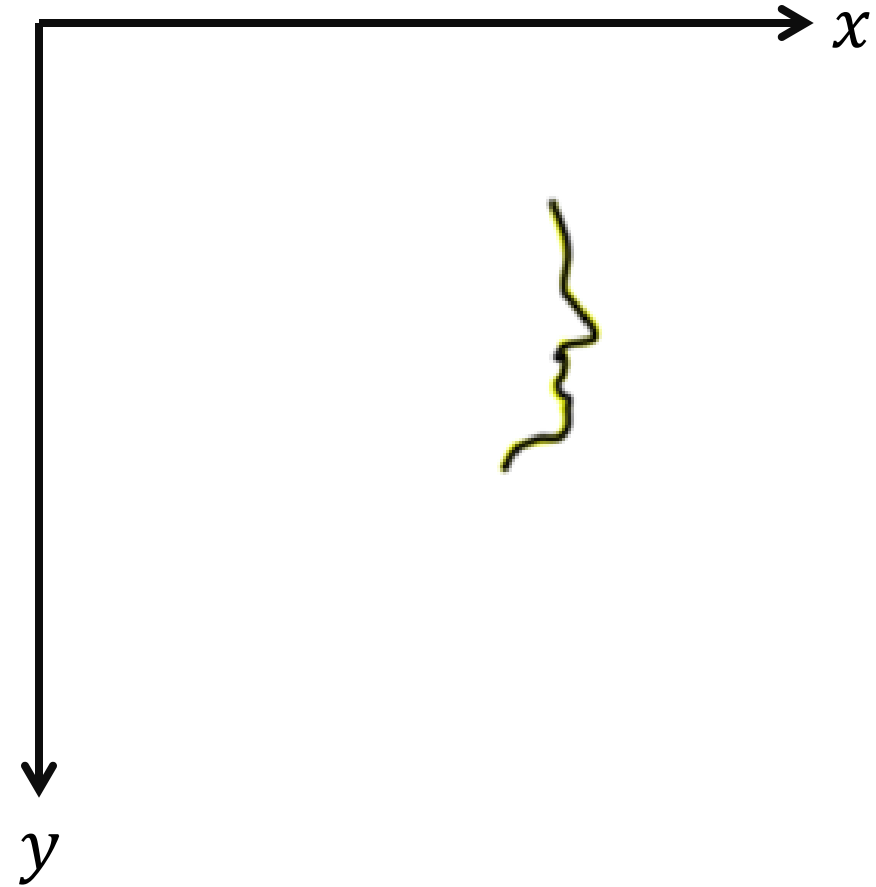


Image rows (or columns) as 2D graphs



Paths as curves in 2D



7	13	12	10	10	13	7	12
8	7	36	41	44	35	11	14
12	44	82	83	83	77	44	13
12	38	79	80	80	75	41	13
11	36	81	76	83	82	43	8
13	42	85	83	79	85	38	10
11	9	35	45	39	44	8	14
13	12	10	13	11	10	10	8

.0U

2D curves for image editing and enhancement

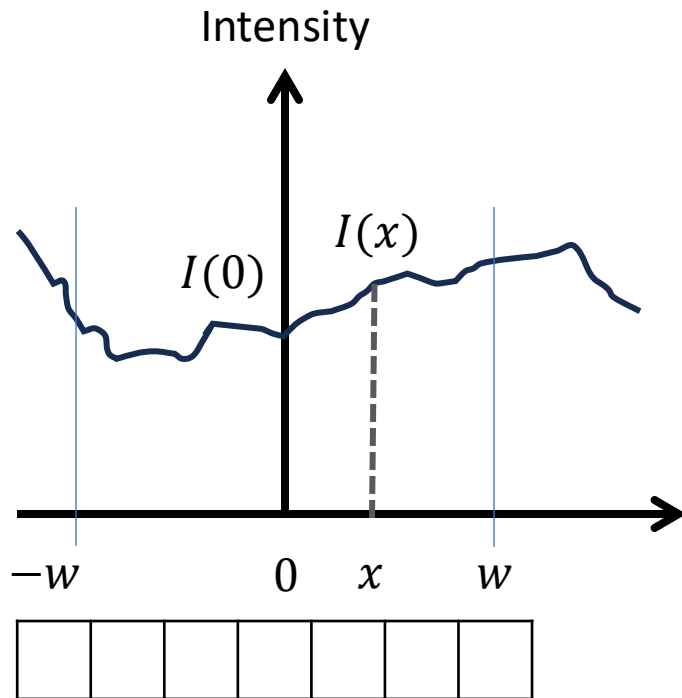
- Object boundaries
- Group of neighboring pixels having the same intensity (isophotes)
- Important tool for image editing and enhancement

Goal

- How do we model 2D curves?
- How do we describe the basic properties of these curves?

Image rows (or columns) as 2D graphs

Polynomial approximation

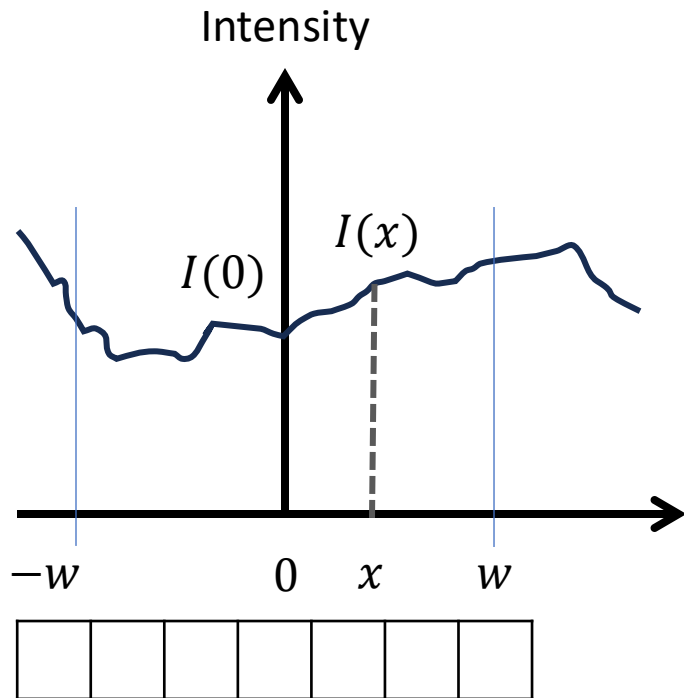


Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = ?$$

Image rows (or columns) as 2D graphs

Polynomial approximation

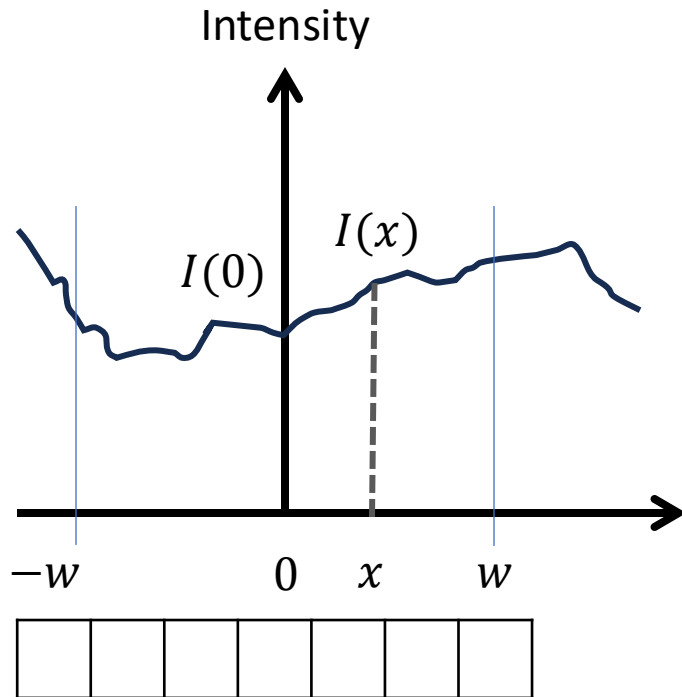


Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = I(0)$$

Image rows (or columns) as 2D graphs

Polynomial approximation

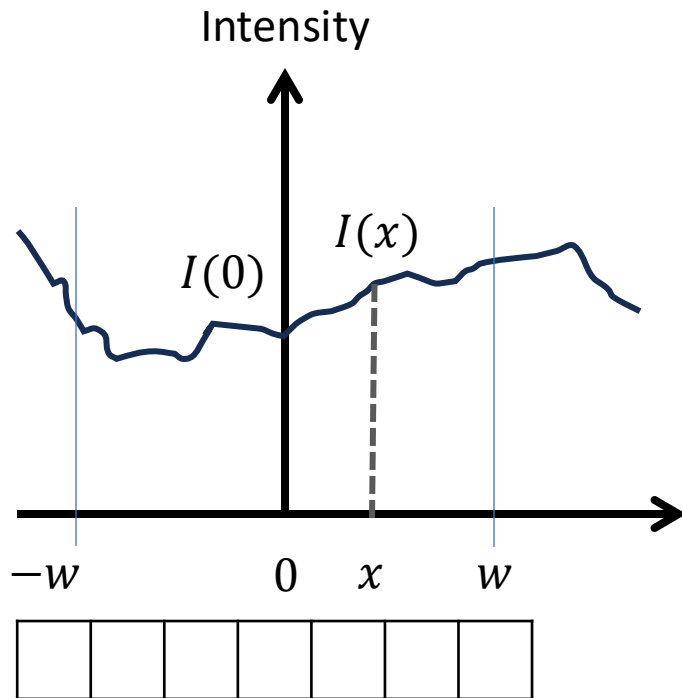


Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = I(0) + xI'(0)$$

Image rows (or columns) as 2D graphs

Polynomial approximation

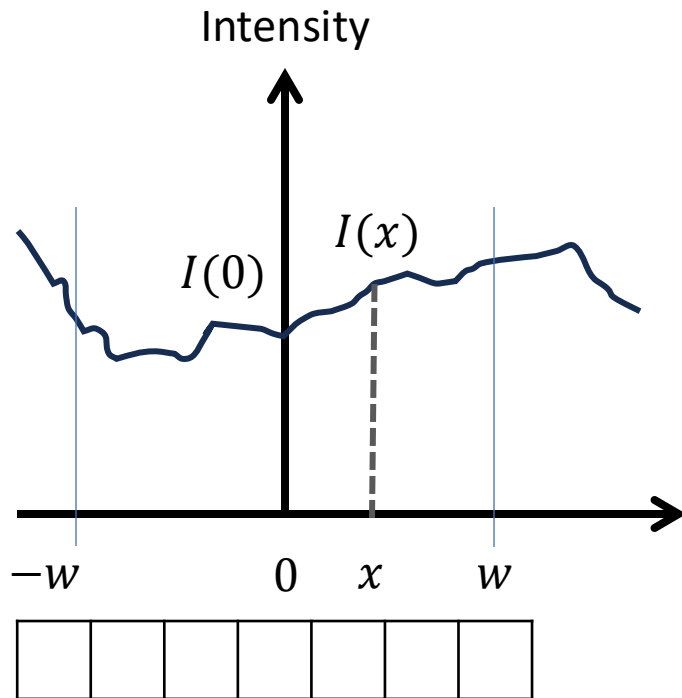


Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = I(0) + xI'(0) + \frac{x^2}{2!}I''(0)$$

Image rows (or columns) as 2D graphs

Polynomial approximation



Taylor series expansion of $I(x)$ near the "patch" center 0

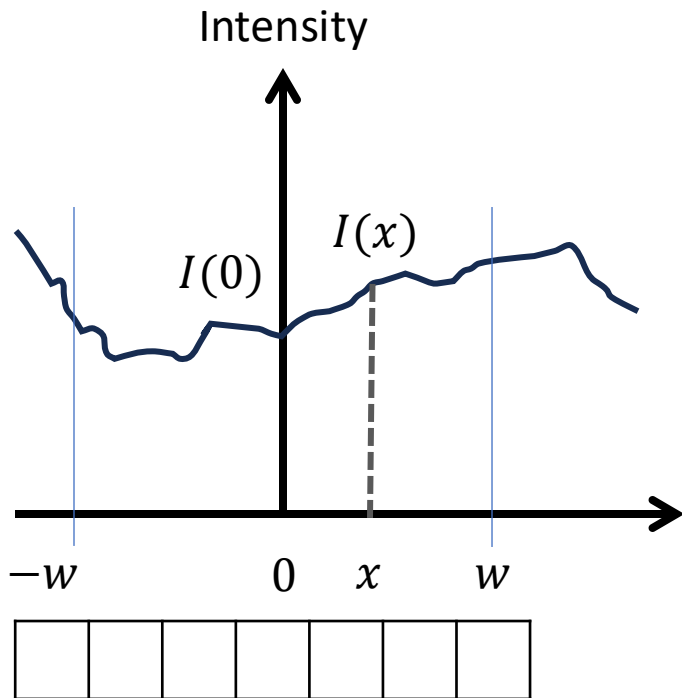
$$I(x) = I(0) + xI'(0) + \frac{x^2}{2!}I''(0) + \frac{x^3}{3!}I'''(0) + \dots + \frac{x^n}{n!}I^{(n)} + R_{n+1}(x)$$

The residual $R_{n+1}(x)$ satisfies:

$$\lim_{x \rightarrow 0} R_{n+1}(x) = 0$$

Image rows (or columns) as 2D graphs

Polynomial approximation



Taylor series expansion of $I(x)$ near the “patch” center 0

$$I(x) = I(0) + xI'(0) + \frac{x^2}{2!}I''(0) + \frac{x^3}{3!}I'''(0) + \dots + \frac{x^n}{n!}I^{(n)} + R_{n+1}(x)$$

Nth order approximation

For a given x , approximation depends on $(n + 1)$ constants corresponding to the intensity derivative at the patch origin.

Image rows (or columns) as 2D graphs

Polynomial approximation

Taylor series expansion of $I(x)$ near the “patch” center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2I''(0) + \frac{1}{6}x^3I'''(0) + \dots + \frac{1}{n!}x^nI^{(n)}(0)$$

Re-write in matrix form

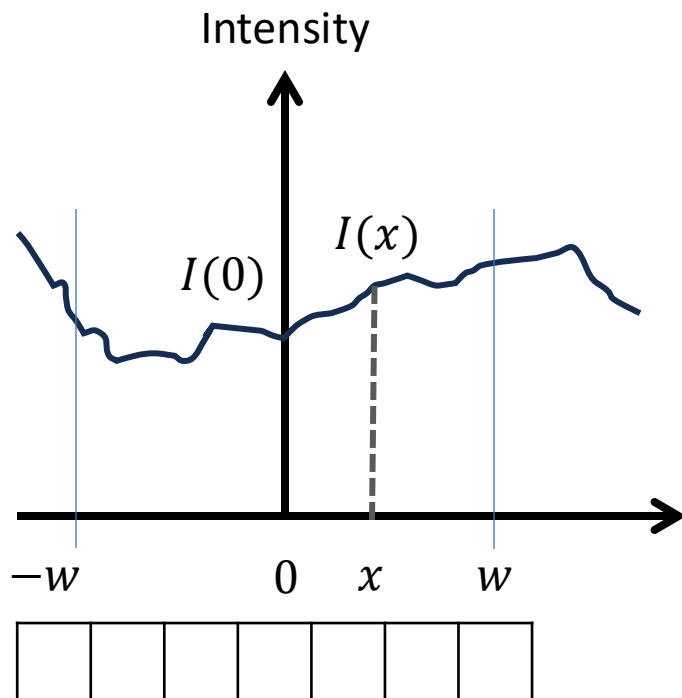
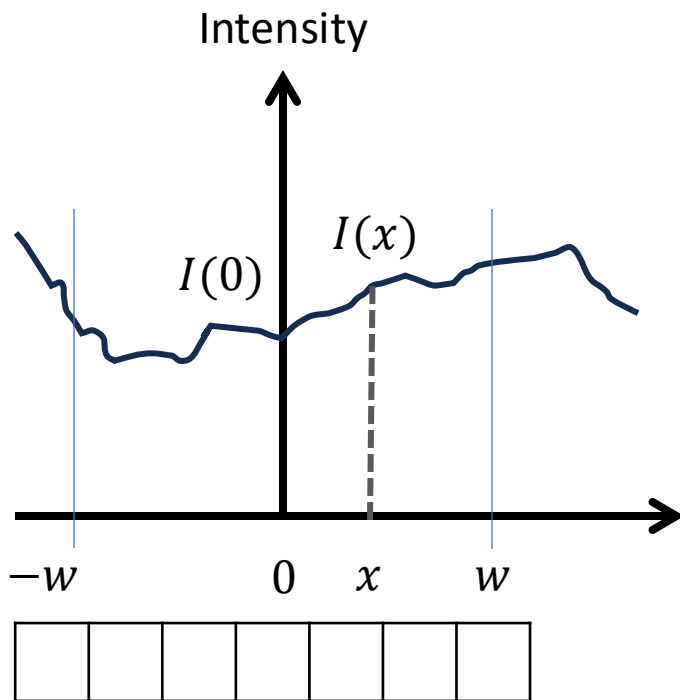


Image rows (or columns) as 2D graphs

Polynomial approximation



Taylor series expansion of $I(x)$ near the “patch” center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2I''(0) + \frac{1}{6}x^3I'''(0) + \dots + \frac{1}{n!}x^nI^{(n)}(0)$$

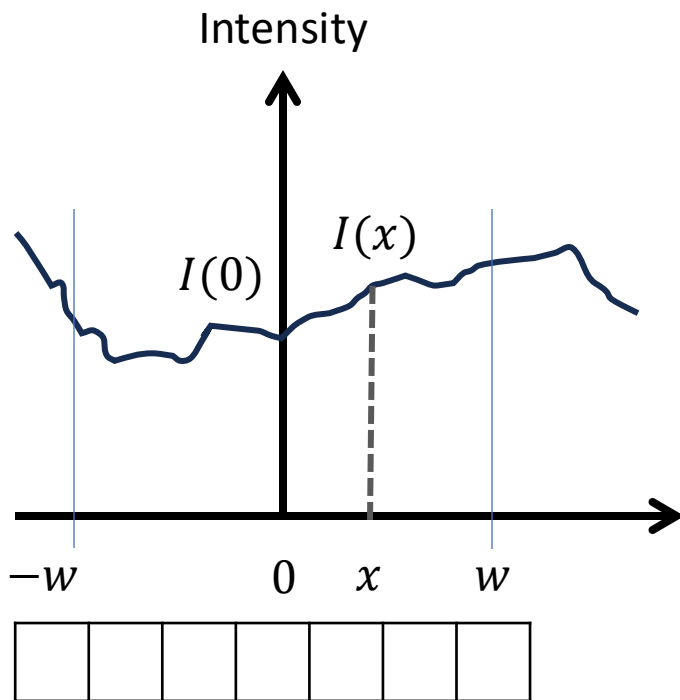
Re-write in matrix form

$$I(x) \approx \begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \dots & \frac{1}{n!}x^n \end{bmatrix} \begin{bmatrix} I(0) \\ I'(0) \\ I''(0) \\ I'''(0) \\ \vdots \\ I^{(n)} \end{bmatrix}$$

For notational simplicity, let's refer the vector of intensity and its derivatives as \mathbf{d}

Image rows (or columns) as 2D graphs

Polynomial approximation



Taylor series expansion of $I(x)$ near the “patch” center 0

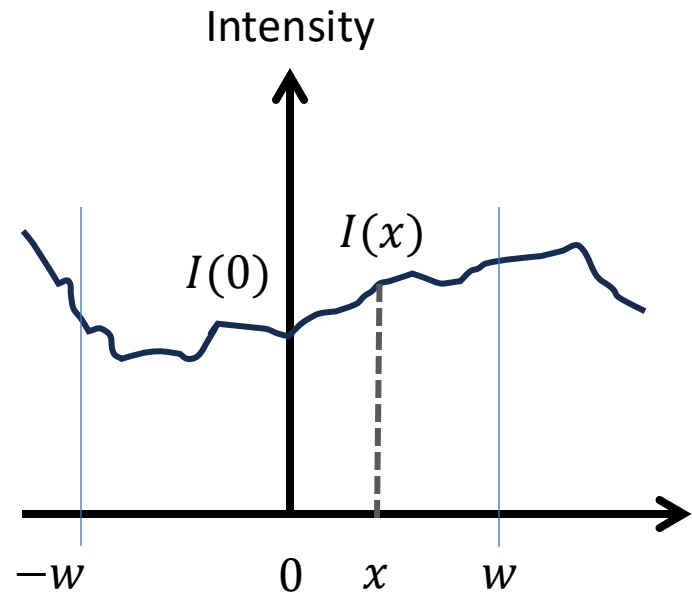
$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2I''(0) + \frac{1}{6}x^3I'''(0) + \dots + \frac{1}{n!}x^nI^{(n)}(0)$$

Practice Question

Show the 1st and 2nd order approximations

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



I_1	I_2	I_3	I_4	I_5	I_6	I_7	Pixel intensities
-------	-------	-------	-------	-------	-------	-------	-------------------

-3 -2 -1 0 1 2 3 Pixel locations

Notice we refer to pixel locations w.r.t. the patch center which sits at location 0

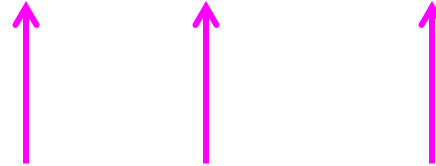
Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities

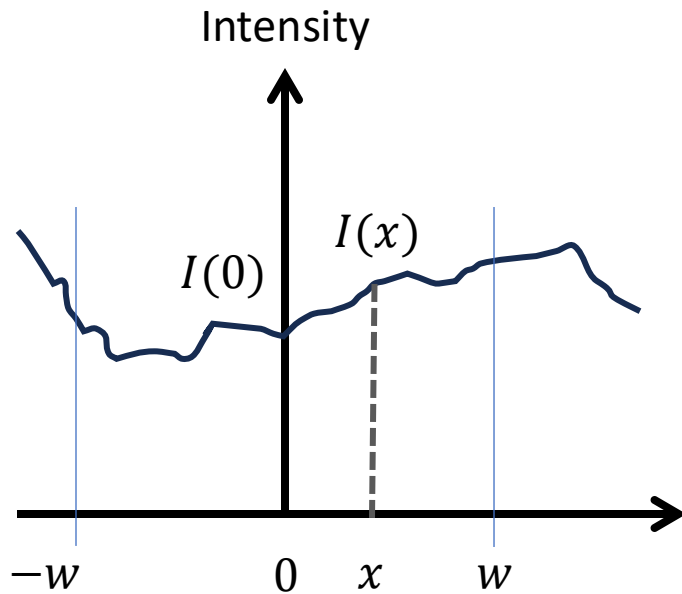
Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2I''(0)$$



Unknowns



I_1	I_2	I_3	I_4	I_5	I_6	I_7
-------	-------	-------	-------	-------	-------	-------

Pixel intensities

-3 -2 -1 0 1 2 3

Pixel locations

Notice we refer to pixel locations w.r.t. the patch center which sits at location 0

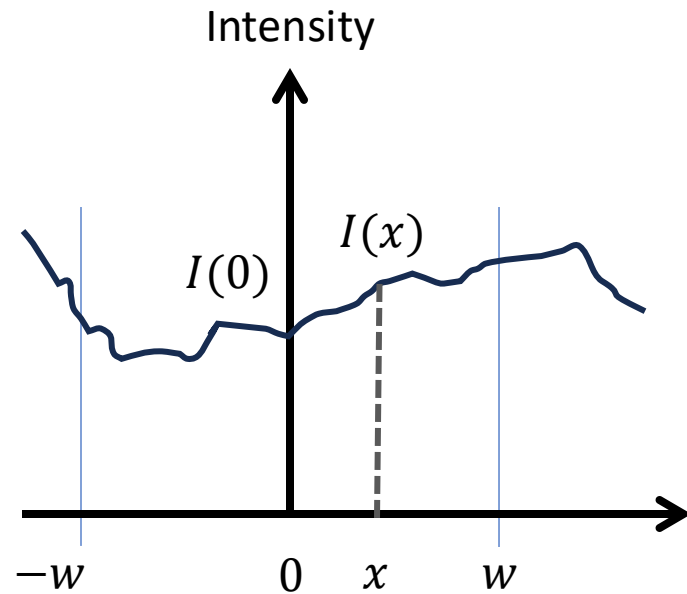
Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities

Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2I''(0)$$



I_1	I_2	I_3	I_4	I_5	I_6	I_7
-------	-------	-------	-------	-------	-------	-------

Pixel intensities

-3 -2 -1 0 1 2 3

Pixel locations

Notice we refer to pixel locations w.r.t. the patch center which sits at location 0

Re-writing one equation per pixel

$$I_1 = I(0) + (-3)I'(0) + \frac{1}{2}(-3)^2I''(0)$$

$$I_2 = I(0) + (-2)I'(0) + \frac{1}{2}(-2)^2I''(0)$$

$$I_3 = I(0) + (-1)I'(0) + \frac{1}{2}(-1)^2I''(0)$$

$$I_4 = I(0) + (0)I'(0) + \frac{1}{2}(0)^2I''(0)$$

$$I_5 = I(0) + (1)I'(0) + \frac{1}{2}(1)^2I''(0)$$

$$I_6 = I(0) + (2)I'(0) + \frac{1}{2}(2)^2I''(0)$$

$$I_7 = I(0) + (3)I'(0) + \frac{1}{2}(3)^2I''(0)$$

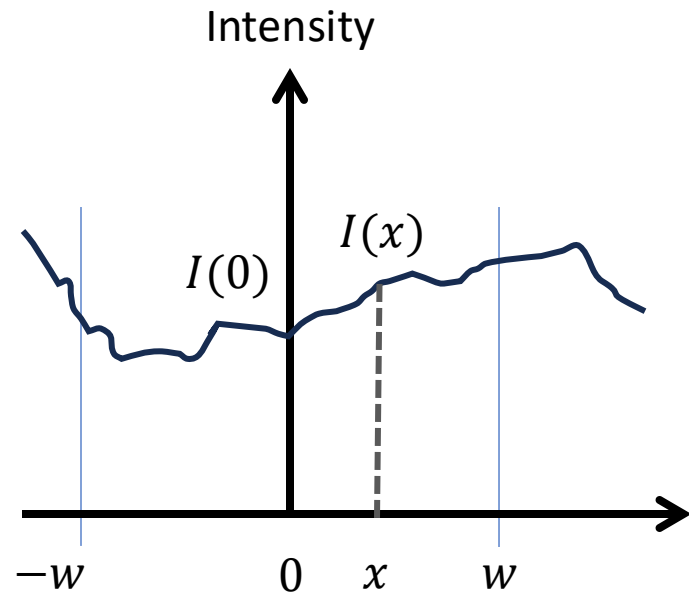
Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities

Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2I''(0)$$



Re-write in matrix form and solve using least squares

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9/2 \\ 1 & -2 & 4/2 \\ 1 & -1 & 1/2 \\ 1 & 0 & 0 \\ 1 & 1 & 1/2 \\ 1 & 2 & 4/2 \\ 1 & 3 & 9/2 \end{bmatrix} \begin{bmatrix} I(0) \\ I'(0) \\ I''(0) \end{bmatrix}$$

Unknowns

I_1	I_2	I_3	I_4	I_5	I_6	I_7	Pixel intensities
-------	-------	-------	-------	-------	-------	-------	-------------------

-3 -2 -1 0 1 2 3 Pixel locations

Notice we refer to pixel locations w.r.t. the patch center which sits at location 0

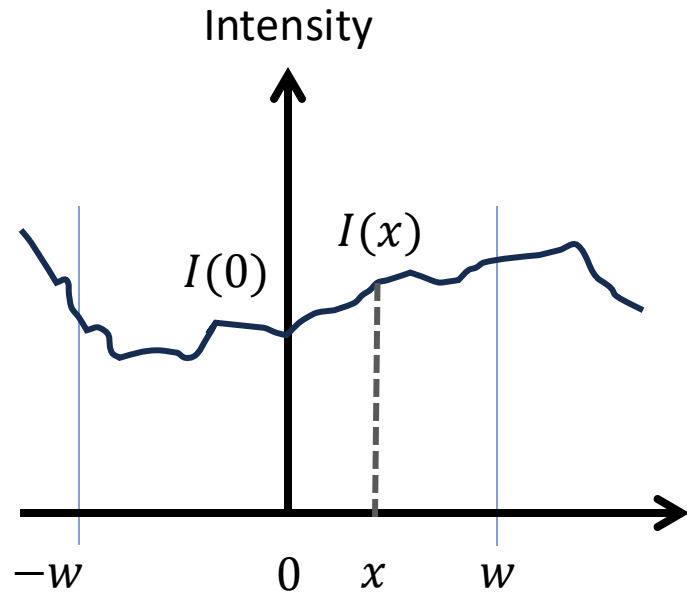
Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities

Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2I''(0)$$



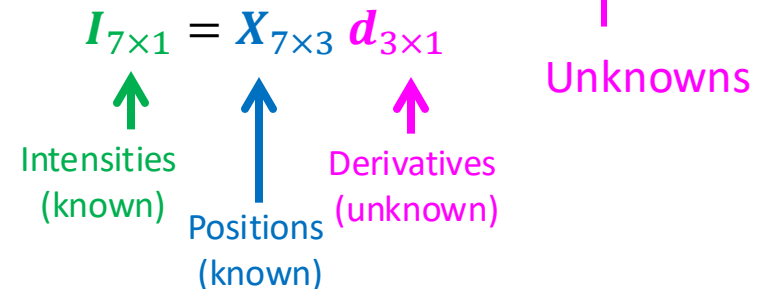
Re-write in matrix form and solve using least squares

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9/2 \\ 1 & -2 & 4/2 \\ 1 & -1 & 1/2 \\ 1 & 0 & 0 \\ 1 & 1 & 1/2 \\ 1 & 2 & 4/2 \\ 1 & 3 & 9/2 \end{bmatrix} \begin{bmatrix} I(0) \\ I'(0) \\ I''(0) \end{bmatrix}$$

I_1	I_2	I_3	I_4	I_5	I_6	I_7	Pixel intensities
-------	-------	-------	-------	-------	-------	-------	-------------------

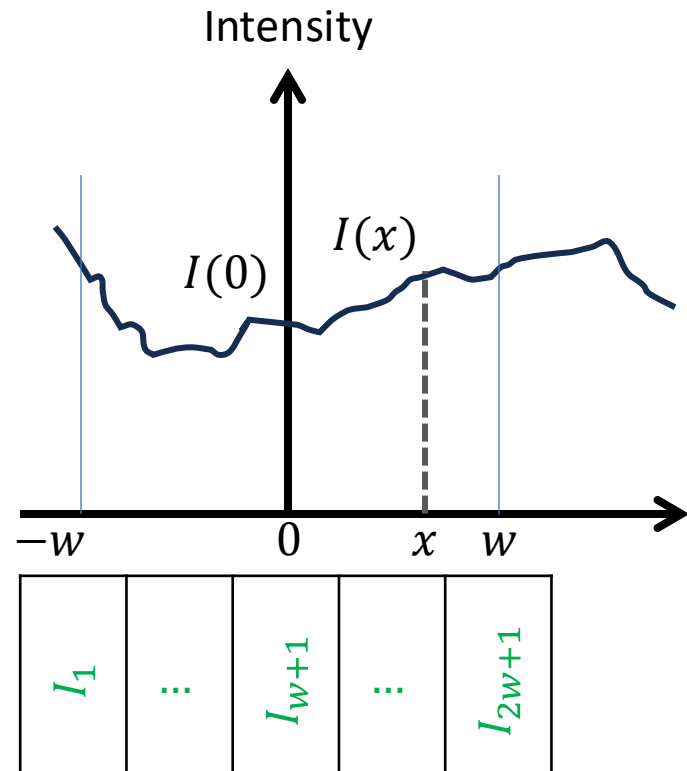
-3 -2 -1 0 1 2 3 Pixel locations

Notice we refer to pixel locations w.r.t. the patch center which sits at location 0



Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree n

Use n th order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2I''(0) + \frac{1}{6}x^3I'''(0) + \dots + \frac{1}{n!}x^nI^{(n)}(0)$$



$(n + 1)$ Unknowns

Observation

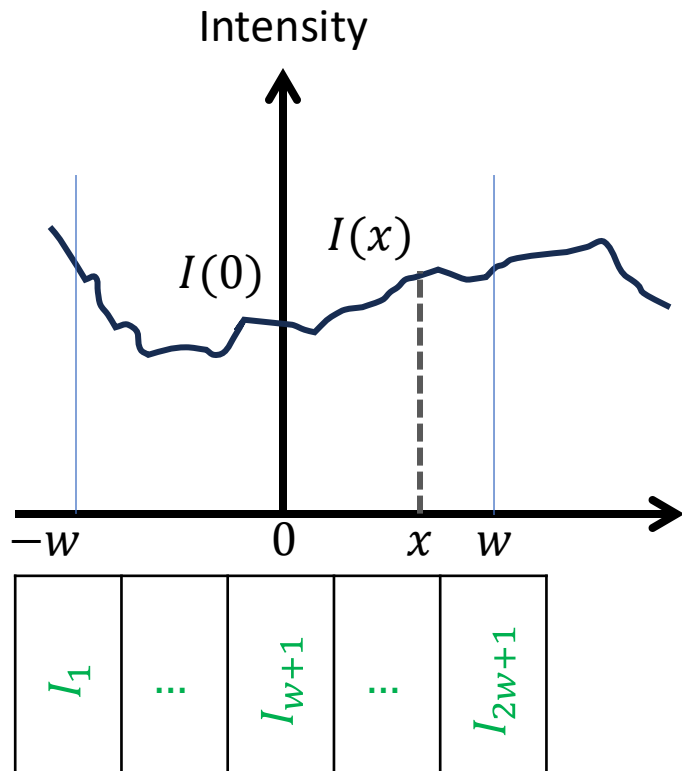
A $(2w + 1)$ -patch gives $2w + 1$ equations.

Conclusion

For a patch of size $(2w + 1)$, it is only possible to fit a polynomial of degree $2w$.

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree n

Use n th order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2I''(0) + \frac{1}{6}x^3I'''(0) + \dots + \frac{1}{n!}x^nI^{(n)}(0)$$



$(n + 1)$ Unknowns

$$I_{(2w+1) \times 1} = X_{(2w+1) \times n} d_{n \times 1}$$

↑
Intensities
(known)

↑
Positions
(known)

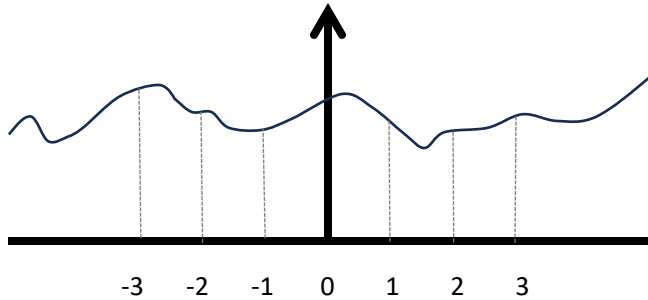
↑
Derivatives
(unknown)

Solve this linear system of equations in terms of d minimizes the fit error.

$$\|I - Xd\|^2$$

Solution d is called the *least squares fit*

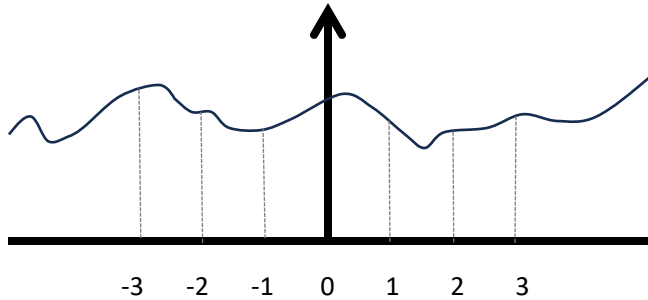
0th order estimation (constant) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [d_0]$$

0th order estimation (constant) of $I(x)$



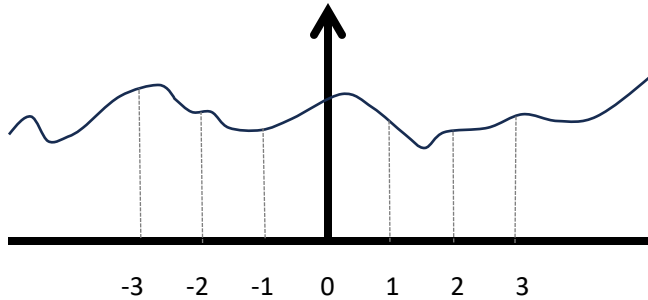
Solution is the mean intensity of the patch

Provides the estimate of intensity of the center of the patch

System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [d_0]$$

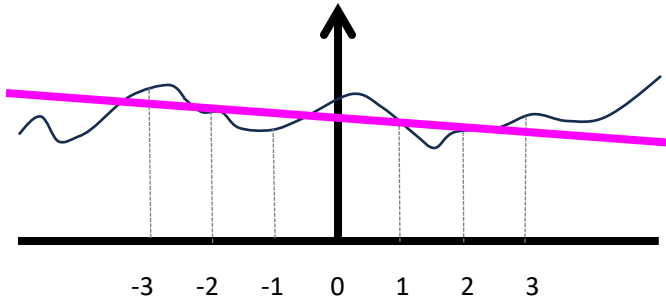
1st order estimation (linear) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$

1st order estimation (linear) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$

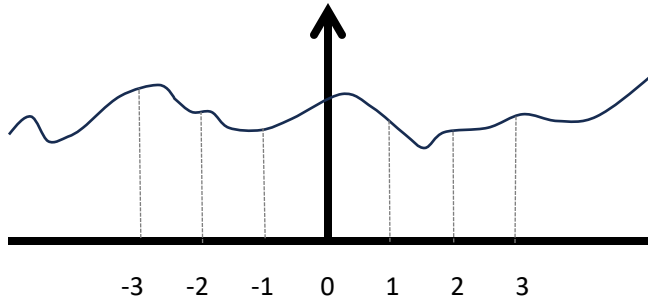
Solution minimizes the sum of vertical distance between the **line** and the image intensities.

Provides the estimate of intensity and its derivative at the patch center

Matrix representation of a line (in 2D)

$$y = b + mx = [1 \quad x] \begin{bmatrix} b \\ m \end{bmatrix}$$

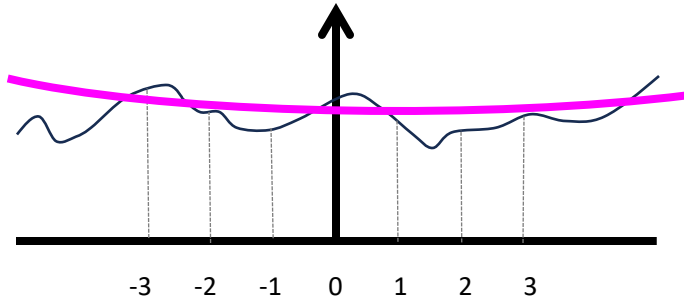
2nd order estimation (quadratic) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9/2 \\ 1 & -2 & 2 \\ 1 & -1 & 1/2 \\ 1 & 0 & 0 \\ 1 & 1 & 1/2 \\ 1 & 2 & 2 \\ 1 & 3 & 9/2 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

2nd order estimation (quadratic) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9/2 \\ 1 & -2 & 2 \\ 1 & -1 & 1/2 \\ 1 & 0 & 0 \\ 1 & 1 & 1/2 \\ 1 & 2 & 2 \\ 1 & 3 & 9/2 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

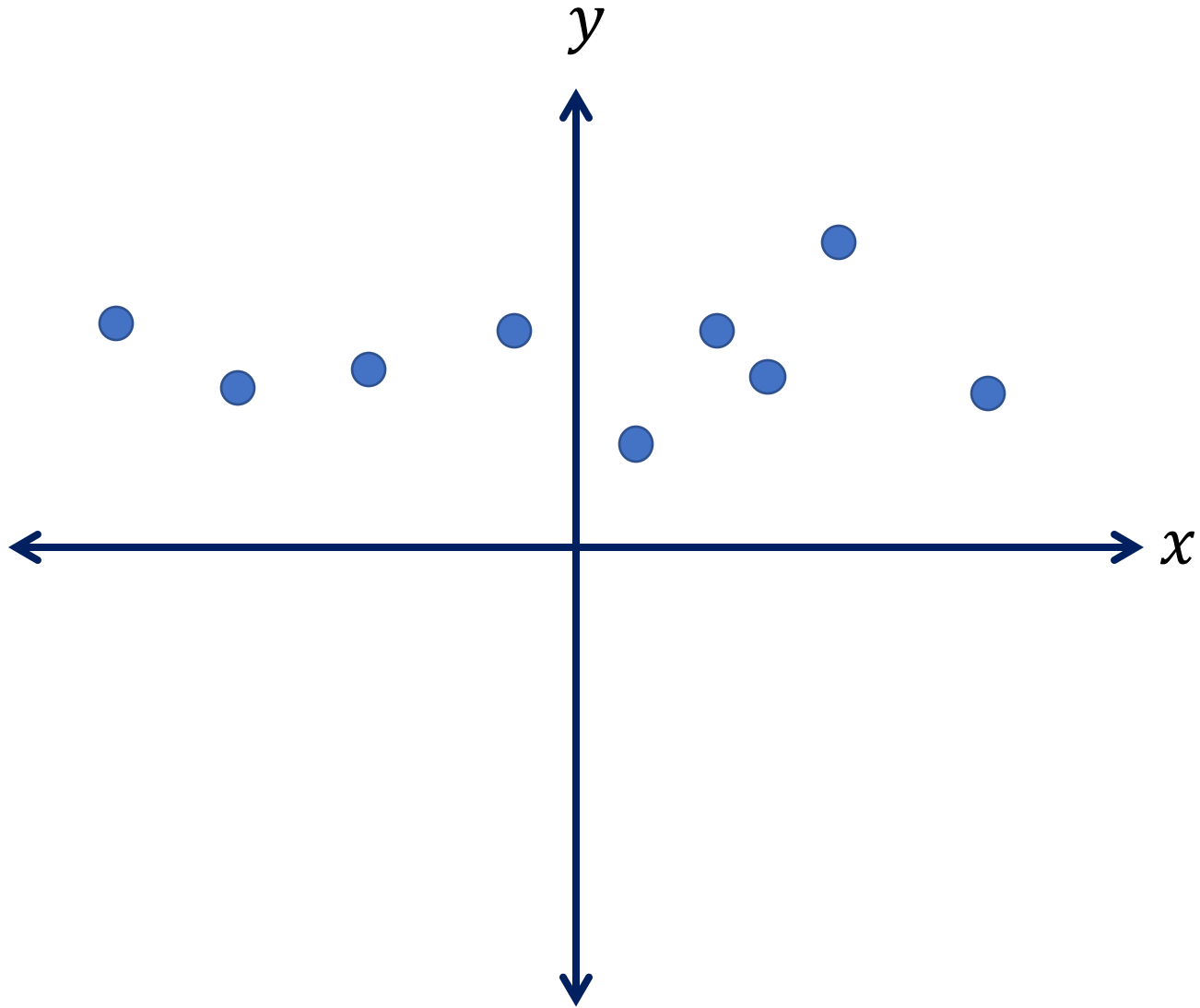
Solution fits a parabola/hyperbola/ellipse to patch intensities

Provides the estimate of intensity and its first and second derivatives at the patch center

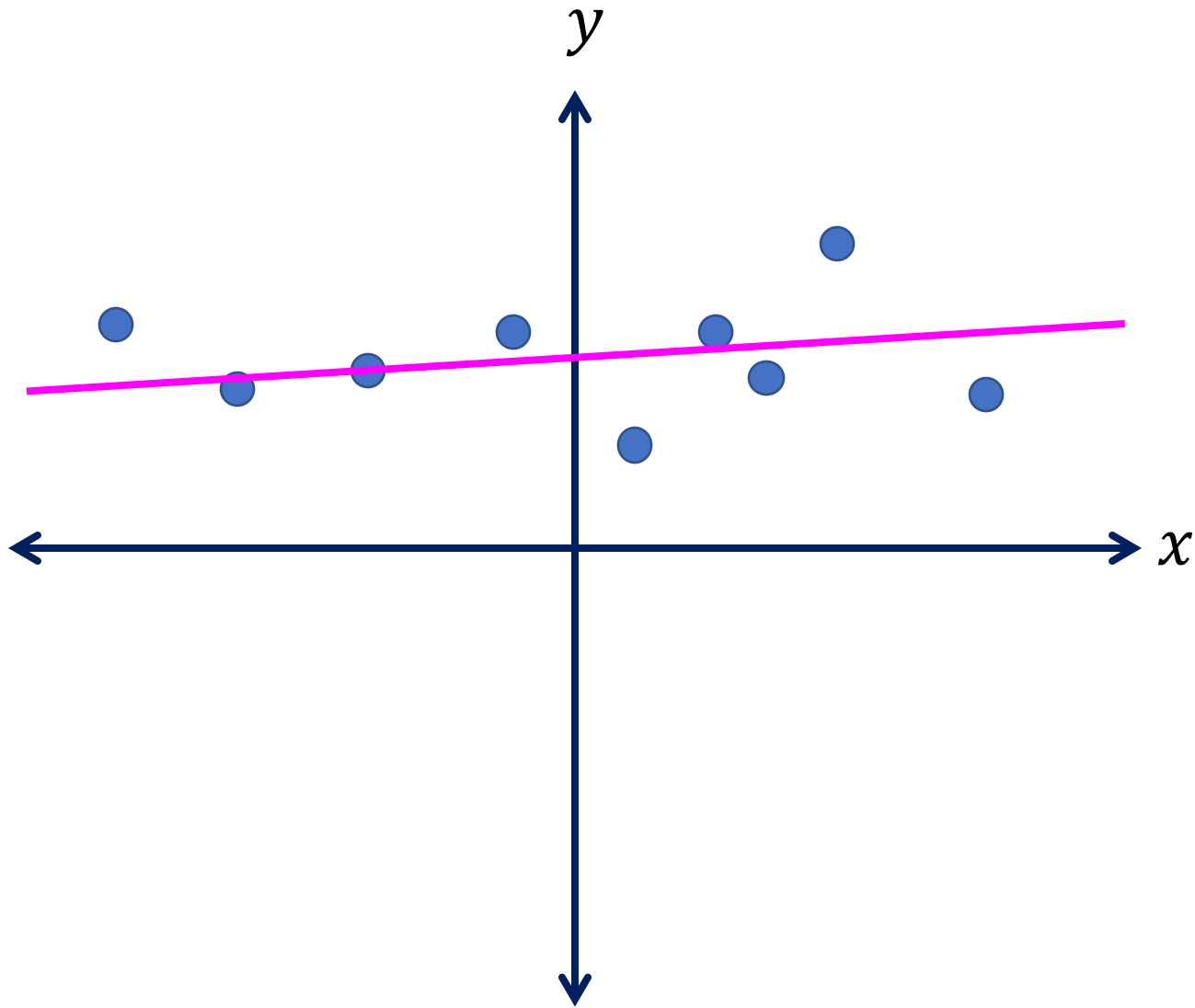
Matrix representation of second order polynomials

$$y = ax^2 + bx + c = [a \quad b \quad c] \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = [x \quad 1] \begin{bmatrix} a \\ \frac{b}{2} \\ c \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Least squares fitting



Least squares fitting



Least squares fitting often use the following notation to represent the system of linear equations

$$Ax = b$$

The solution is

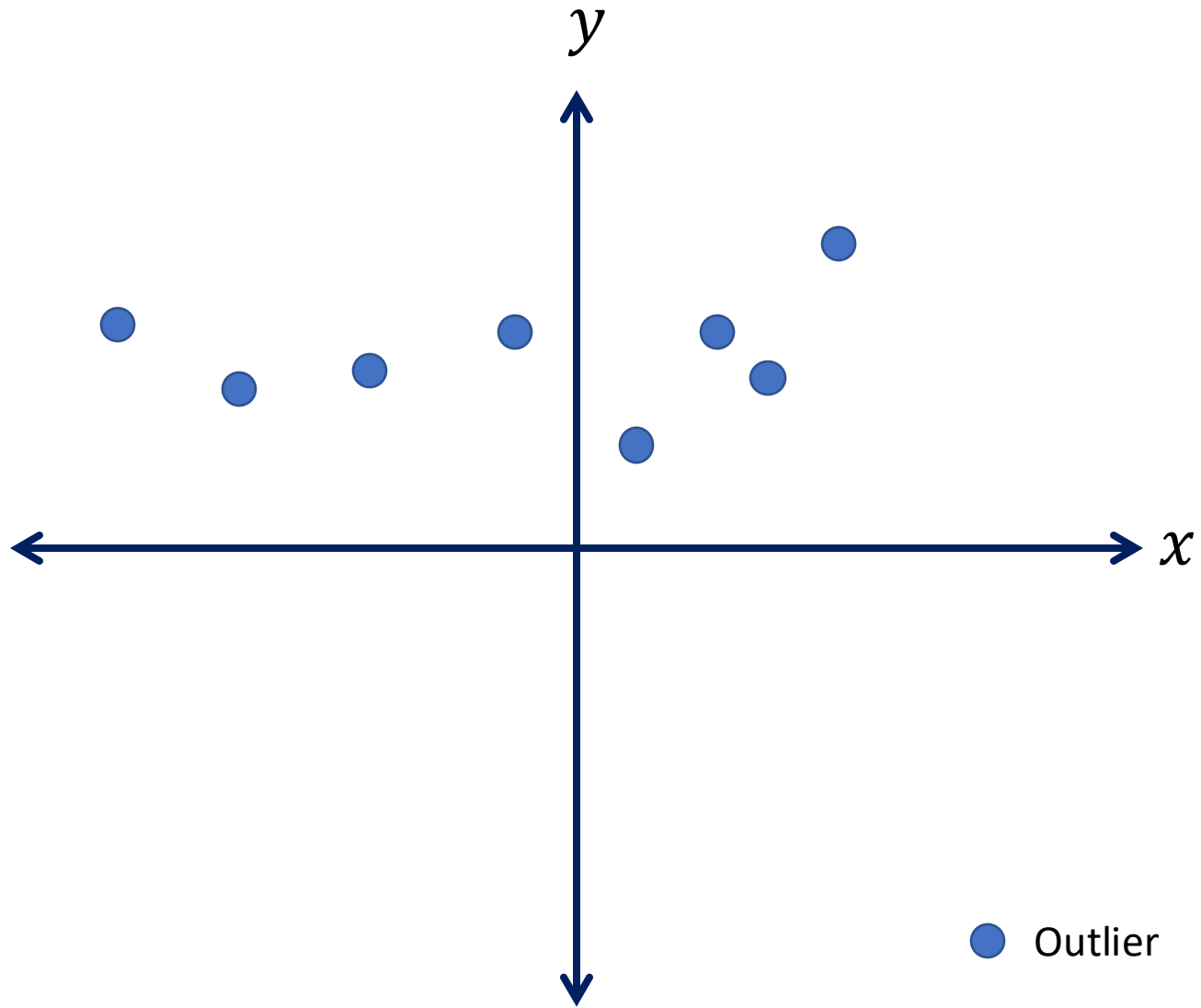
$$x = A^{-1}b$$

where A^{-1} is inverse (or pseudo-inverse) of A .

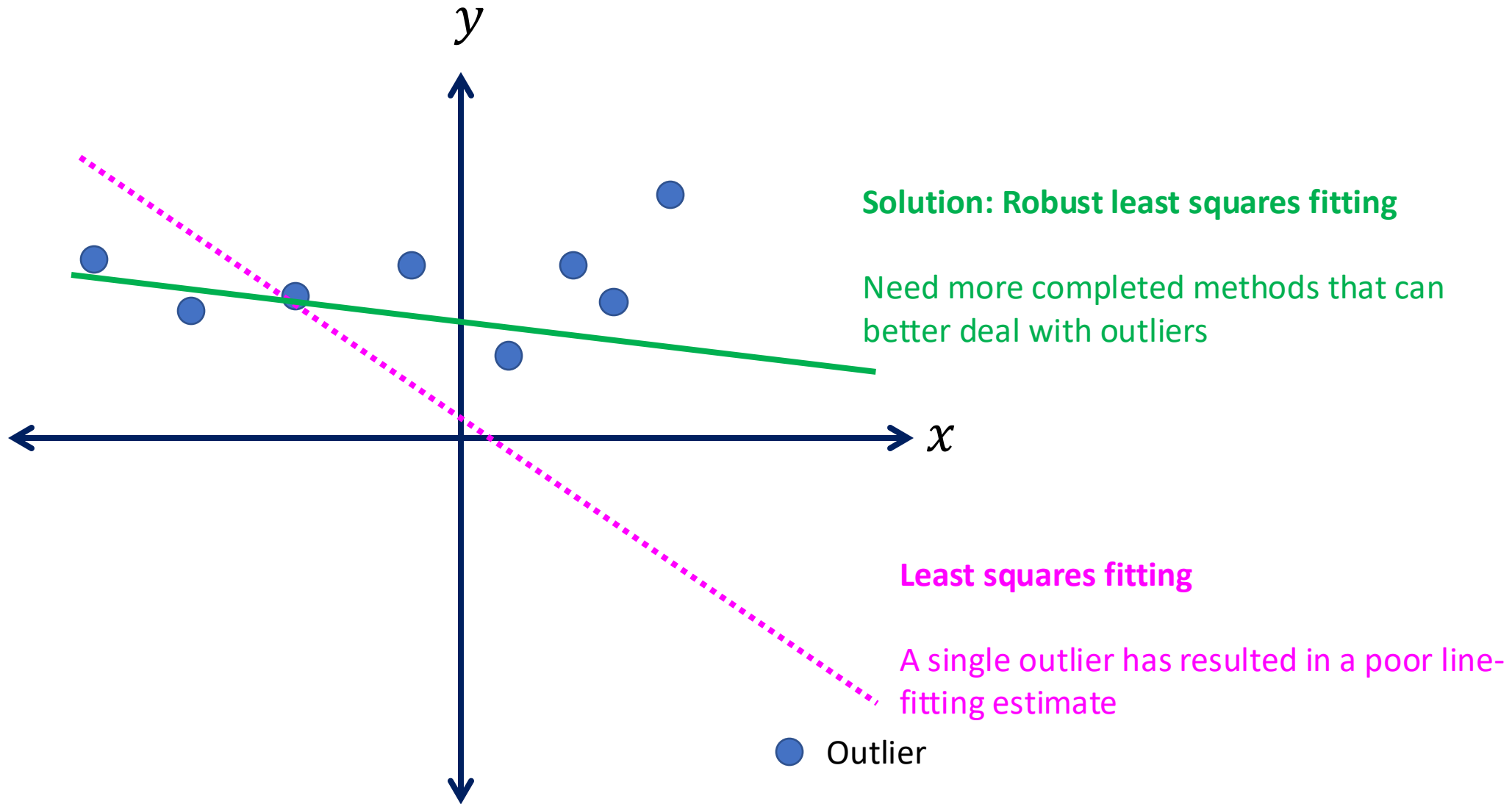
Recall that we need to solve the following system of linear equations when approximating patches with polynomials.

$$\underbrace{I_{(2w+1) \times 1}}_b = \underbrace{X_{(2w+1) \times n}}_A \underbrace{d_{n \times 1}}_x$$

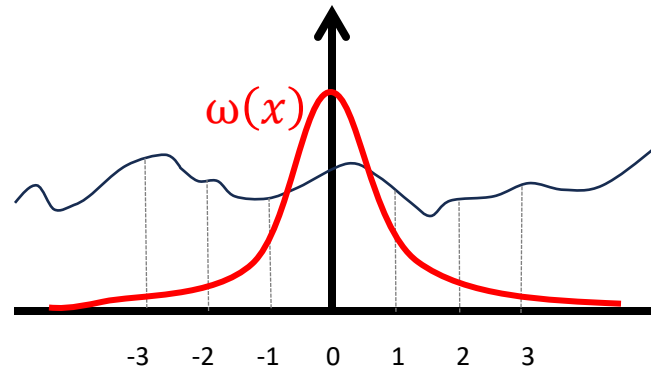
Least squares fitting



Least squares fitting



Weighted least squares estimate of $I(x)$



Give more weight to the pixels near center and less weight to pixels that are far from center,

e.g., $\omega(x) = e^{-x^2}$

Bias our estimate of $I'(0)$ towards the center of the patch.

For patch



The system of linear equations becomes

$$\begin{bmatrix} \omega_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_{2w+1} \end{bmatrix} \mathbf{I}_{(2w+1) \times 1} = \begin{bmatrix} \omega_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_{2w+1} \end{bmatrix} \mathbf{X}_{(2w+1) \times n} \mathbf{d}_{n \times 1}$$

and the solution \mathbf{d} minimizes the norm:

$$\left\| \begin{bmatrix} \omega_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_{2w+1} \end{bmatrix} (\mathbf{I} - \mathbf{X}\mathbf{d}) \right\|^2$$

Estimating image derivatives

- For each row y , define a window of width $2w + 1$ at pixel (i.e., column) x
 - Fit a polynomial (usually of degree 1 or 2)
 - Assign the fitted polynomial's derivatives at location 0 (i.e., center of the patch, or column y in the image space)
 - Slide the window one over, until the end of the row

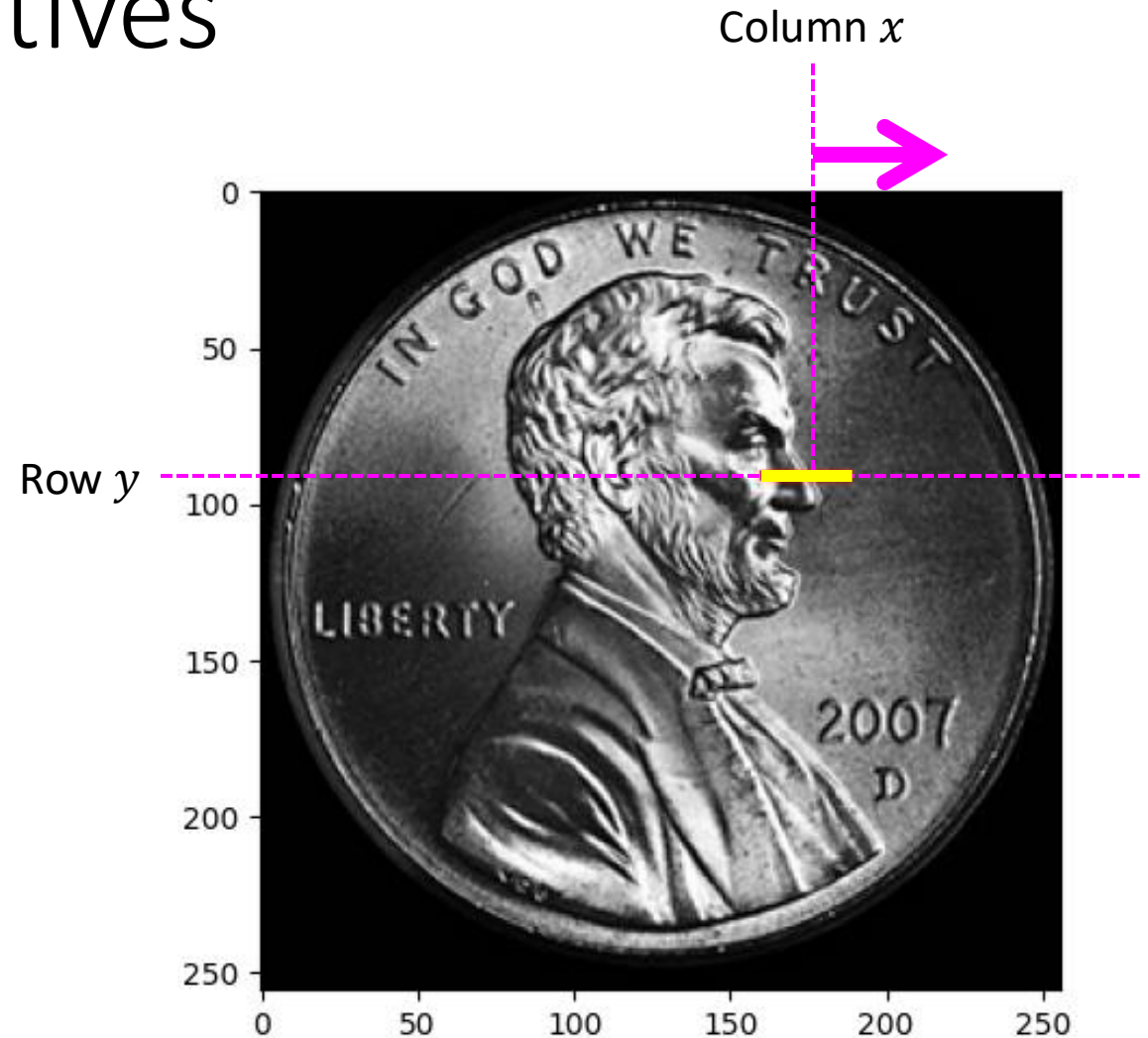
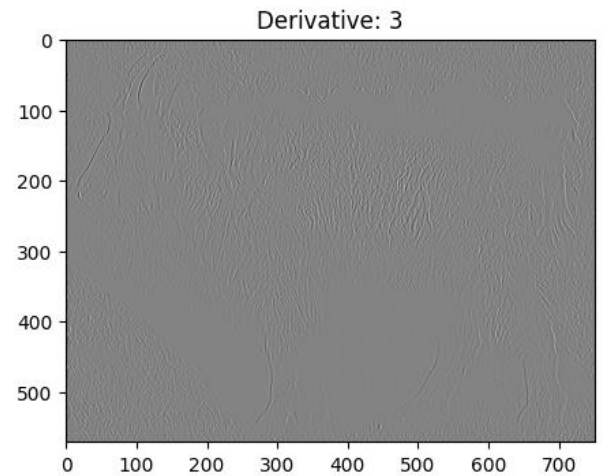
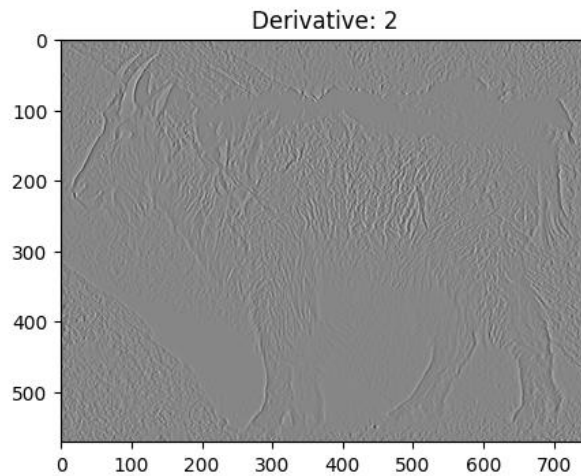
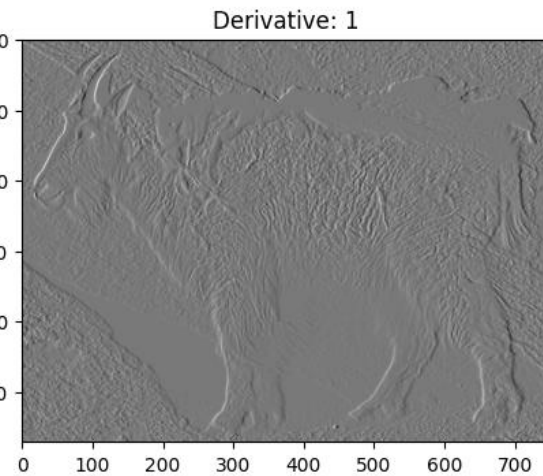
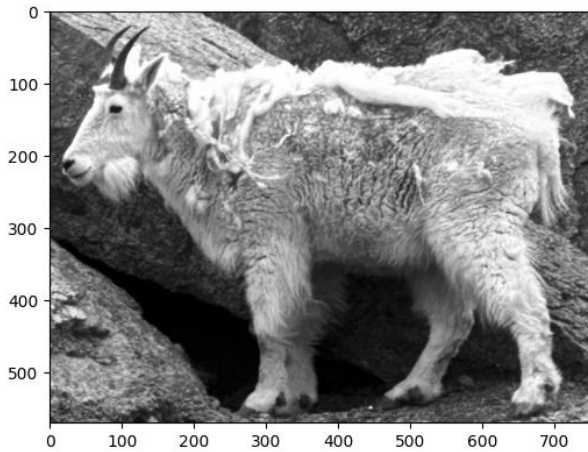


Image derivatives

Fitting a 3rd-order Taylor series using a 5-pixel patch



Summary

- 1D image patches
- Approximating 1D image patches via polynomials
- Computing **image derivatives** via fitting polynomials
- Least squares solution to a system of linear equations
- Weighted least squares