Questions	q1	q2	q3	q4	q5	q6	q7	q8	q9	total
Mean	77.7777778	75.5555556	88.4615385	61.875	75.7142857	56.4102564	47.3484848	64.7058824	51.6666667	50.0649351
Maximum	100	100	100	100	100	100	100	100	100	91.4285714

Polynomial Approximation

Computational Photography (CSCI 3240U)

Faisal Z. Qureshi
http://vclab.science.ontariotechu.ca
Ontario Tech University

Today's lecture

- How to compute image derivatives by fitting polynomials to 1D image patches?
  - Taylor series expansion around a patch center
  - Least square fitting of a system of linear equations

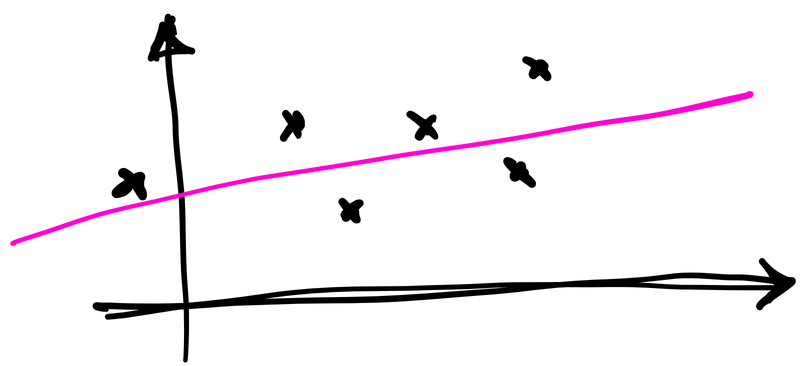
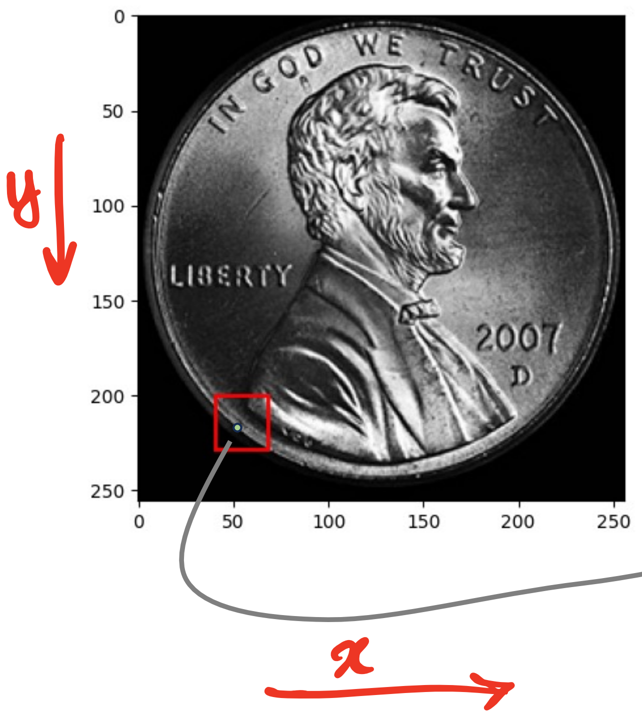
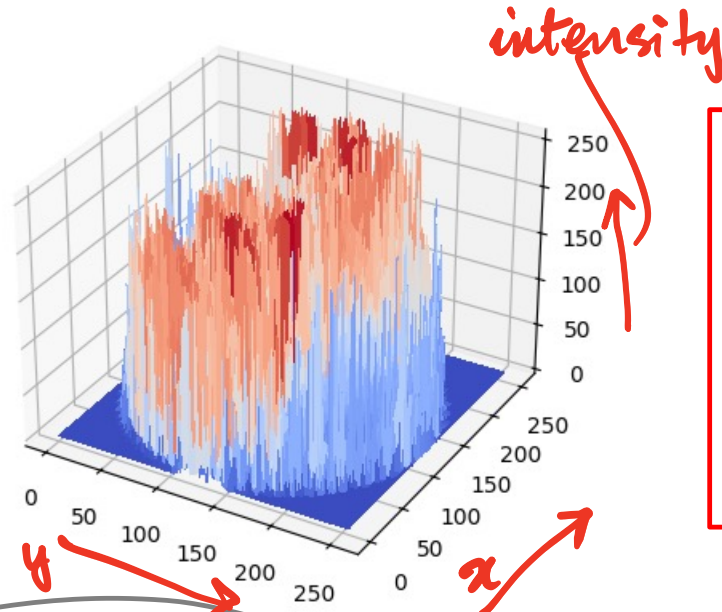


Image as a surface in 3D

Consider a gray-scale image $I(x, y)$ then the height of the surface at (x, y) is $I(x, y)$. The surface passes through the 3D point $(x, y, I(x, y))$.





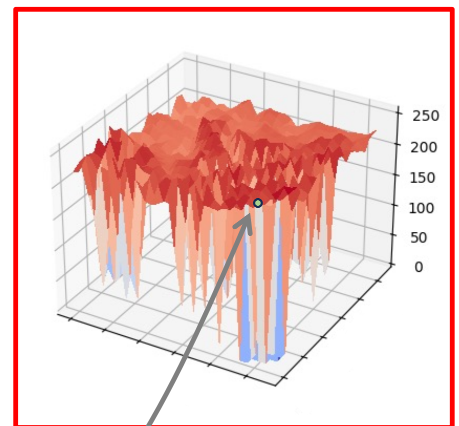
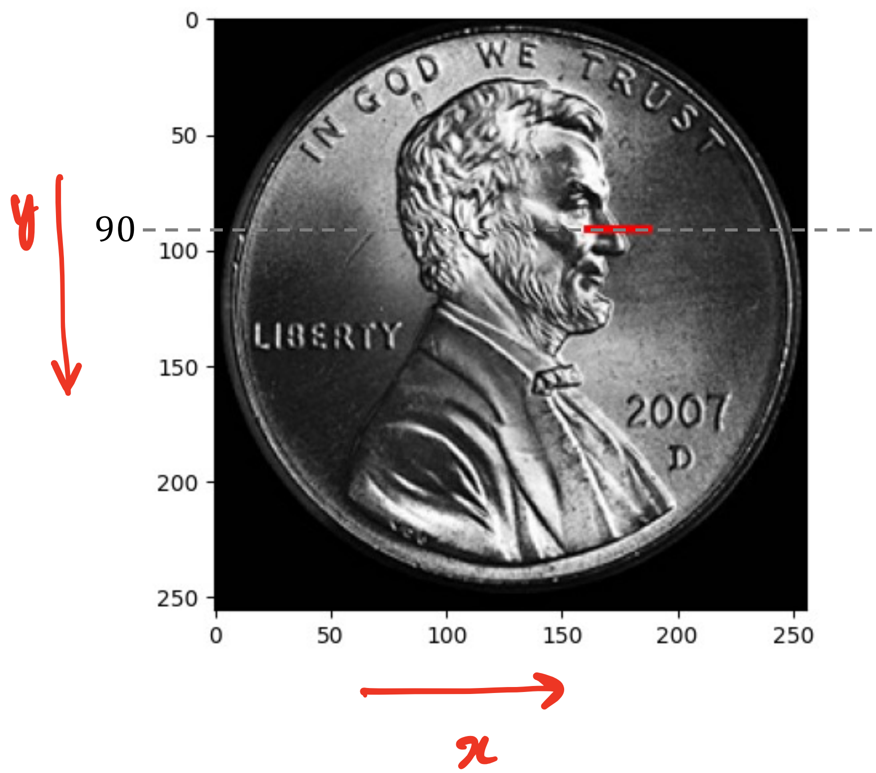
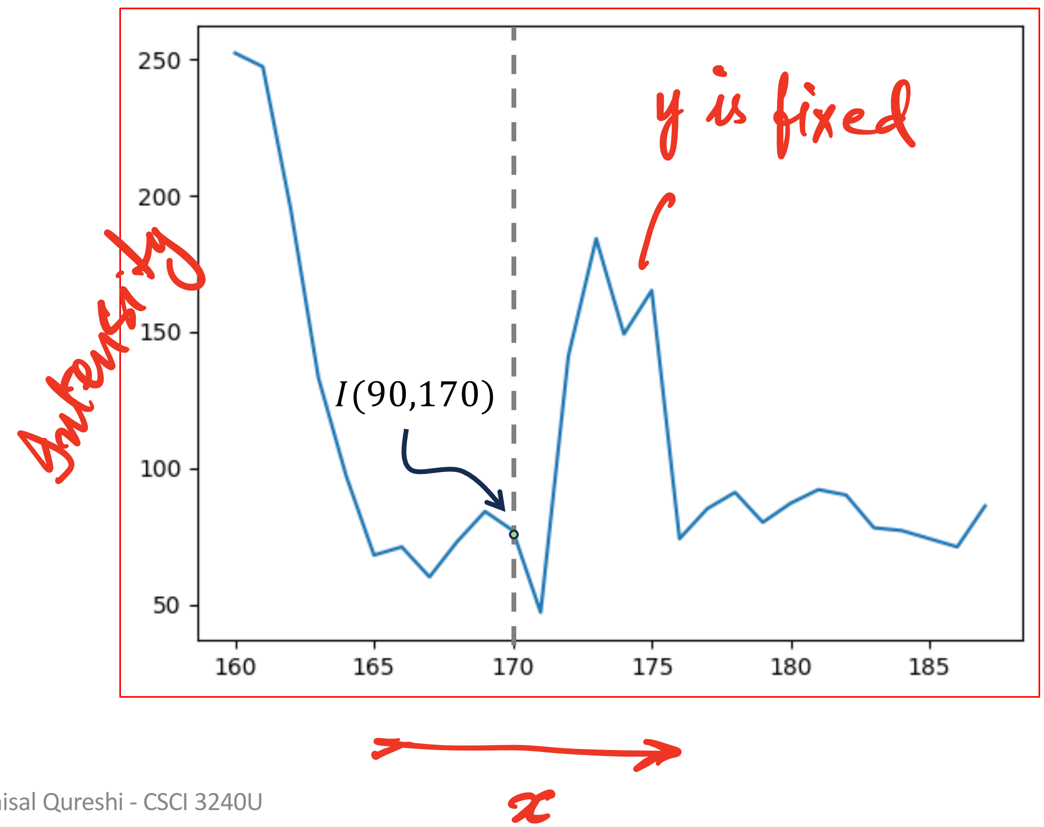
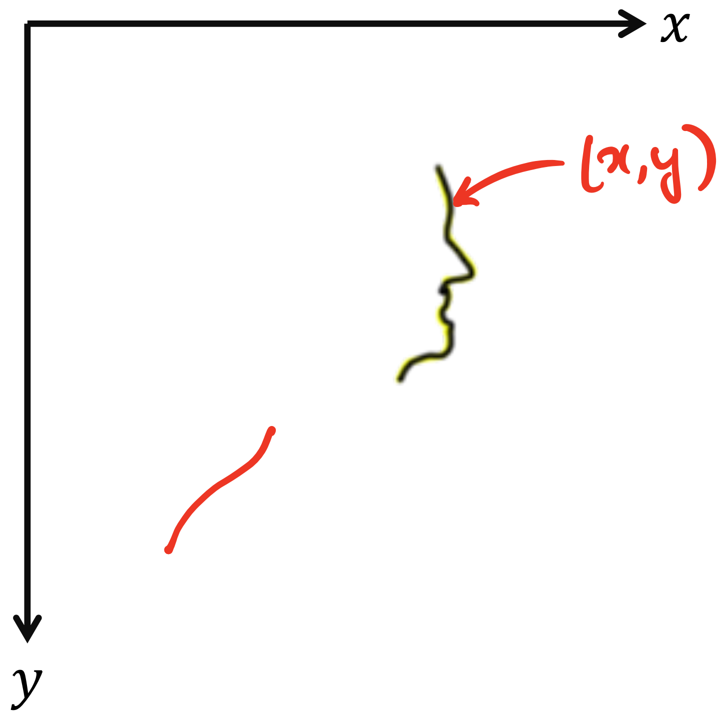


Image rows (or columns) as 2D graphs



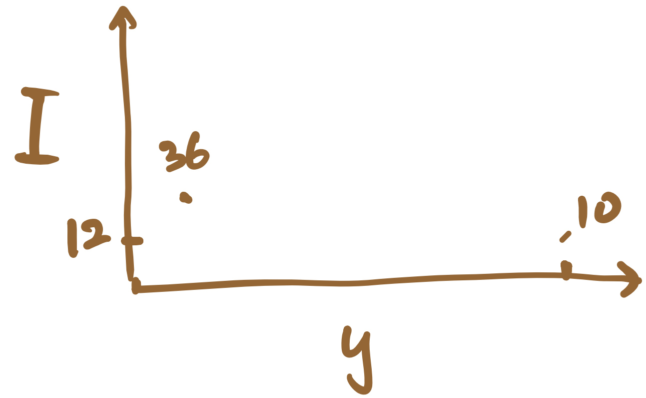


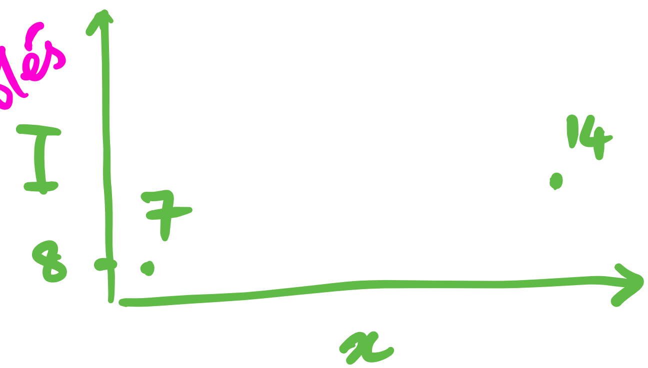
Paths as curves in 2D

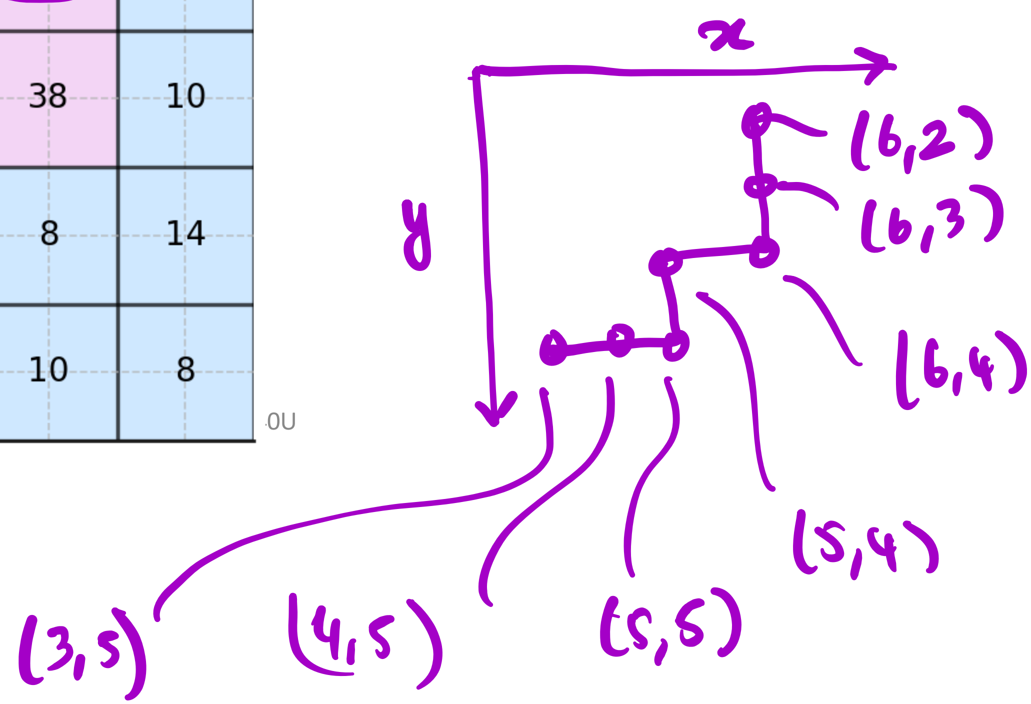


y \ x	0	1	2	3	4	5	6	7
0	7	13	12	10	10	13	7	12
1	8	7	36	41	44	35	11	14
2	12	44	82	83	83	77	44	13
3	12	38	79	80	80	75	41	13
4	11	36	81	76	83	82	43	8
5	13	42	85	83	79	85	38	10
6	11	9	35	45	39	44	8	14
7	13	12	10	13	11	10	10	8

isophotes







2D curves for image editing and enhancement

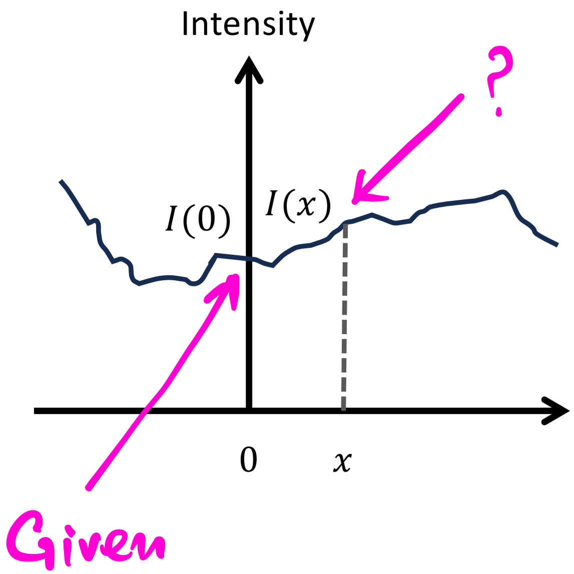
- Object boundaries
- Group of neighboring pixels having the same intensity (isophotes)
- Important tool for image editing and enhancement

Goal

- How do we model 2D curves?
- How do we describe the basic properties of these curves?

Image rows (or columns) as 2D graphs

Polynomial approximation



Taylor series expansion of $I(x)$ near the "patch" center 0

$I(x) = ?$

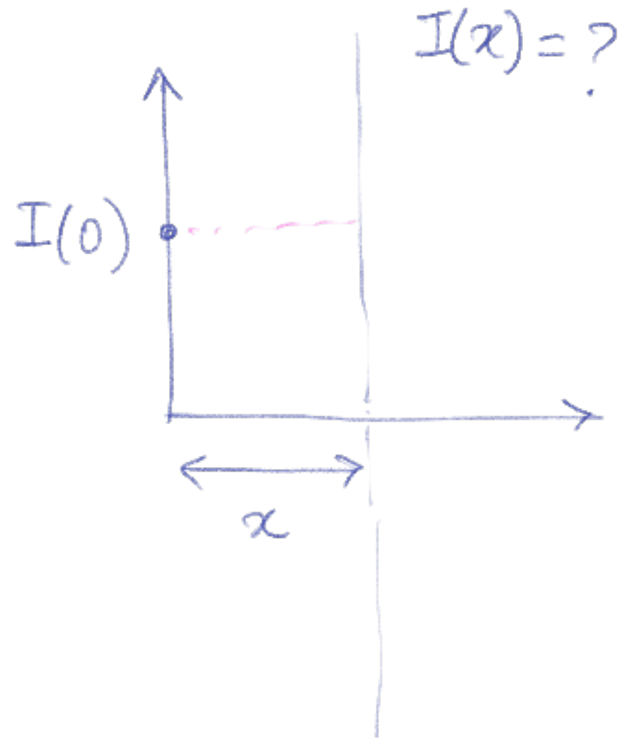
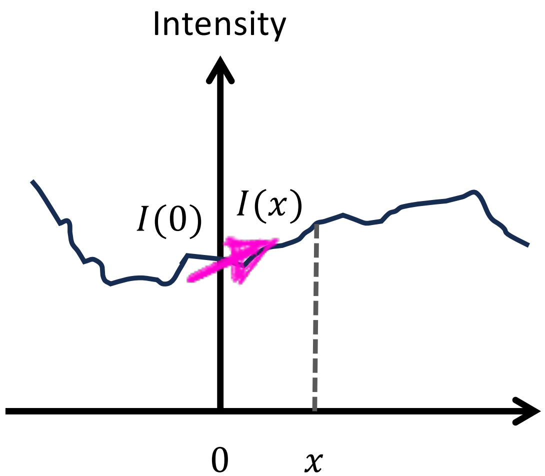


Image rows (or columns) as 2D graphs

Polynomial approximation

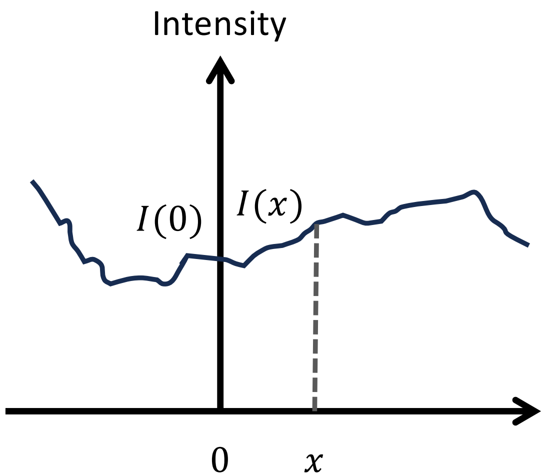


Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = I(0)$$

Image rows (or columns) as 2D graphs

Polynomial approximation



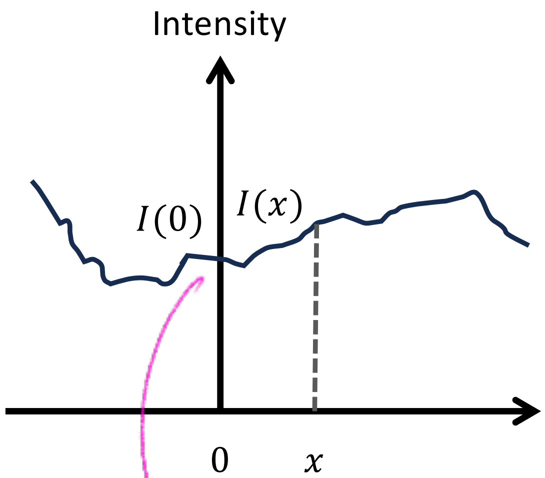
Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = I(0) + xI'(0)$$

First Derivative

Image rows (or columns) as 2D graphs

Polynomial approximation



$I'(0)$

$I''(0)$

Taylor series expansion of $I(x)$ near the "patch" center 0

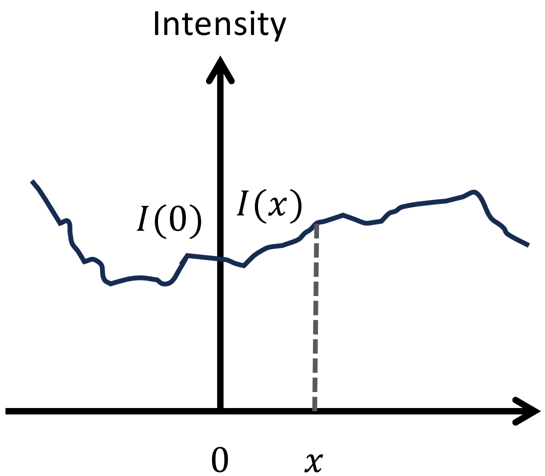
$$I(x) = I(0) + xI'(0) + \frac{x^2}{2!}I''(0)$$

Second derivative

Prediction

Image rows (or columns) as 2D graphs

Polynomial approximation



Taylor series expansion of $I(x)$ near the “patch” center 0

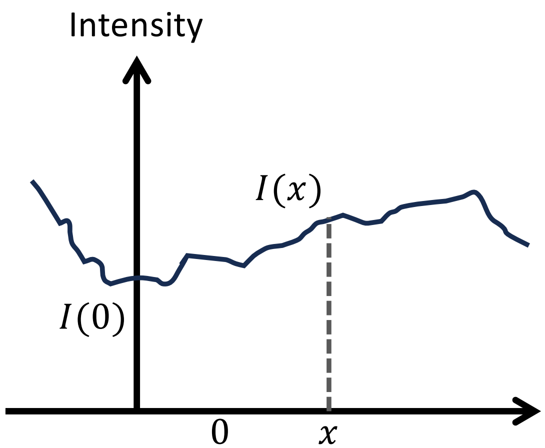
$$I(x) = I(0) + xI'(0) + \frac{x^2}{2!}I''(0) + \frac{x^3}{3!}I'''(0) + \cdots + \frac{x^n}{n!}I^{(n)} + R_{n+1}(x)$$

The residual $R_{n+1}(x)$ satisfies:

$$\lim_{x \to 0} R_{n+1}(x) = 0$$

Image rows (or columns) as 2D graphs

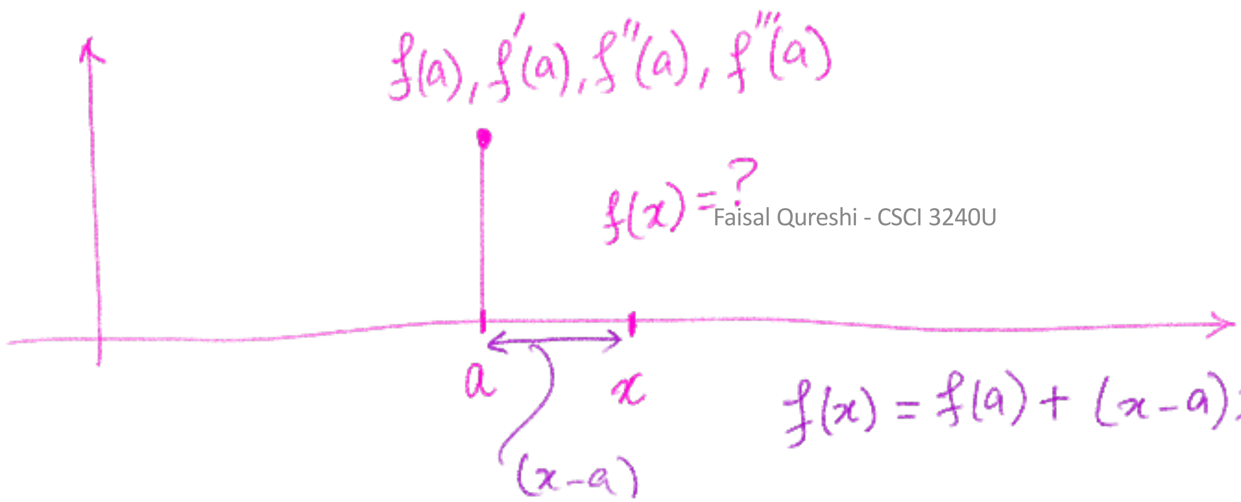
Polynomial approximation



Taylor series expansion of $I(x)$ near the "patch" center 0

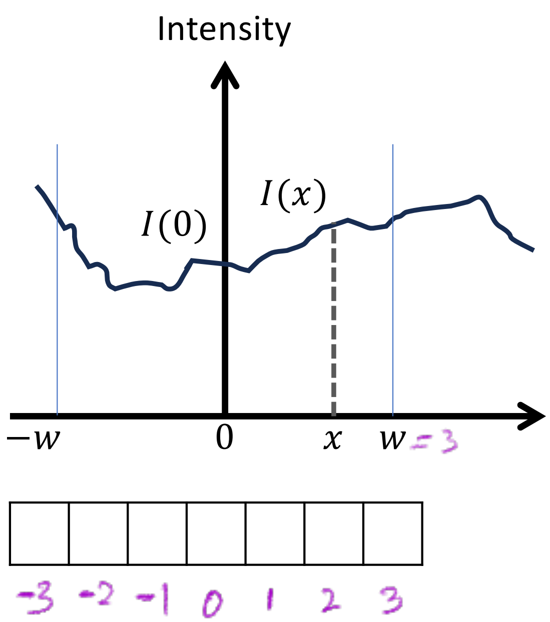
$$\underbrace{I(x) = I(0) + xI'(0) + \frac{x^2}{2!}I''(0) + \frac{x^3}{3!}I'''(0) + \cdots + \frac{x^n}{n!}I^{(n)}}_{\text{Nth order approximation}} + R_{n+1}(x)$$

For a given x, approximation depends on $(n+1)$ constants corresponding to the intensity derivative at the patch origin.



$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a)$$

Polynomial approximation



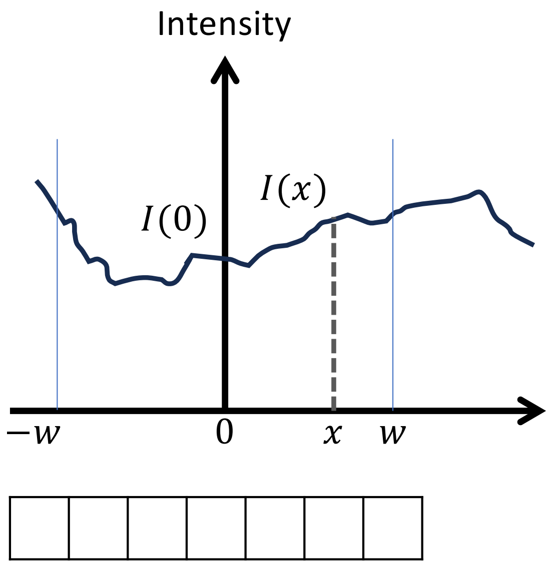
Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Re-write in matrix form

Polynomial approximation

approx



Taylor series expansion of $I(x)$ near the patch center 0

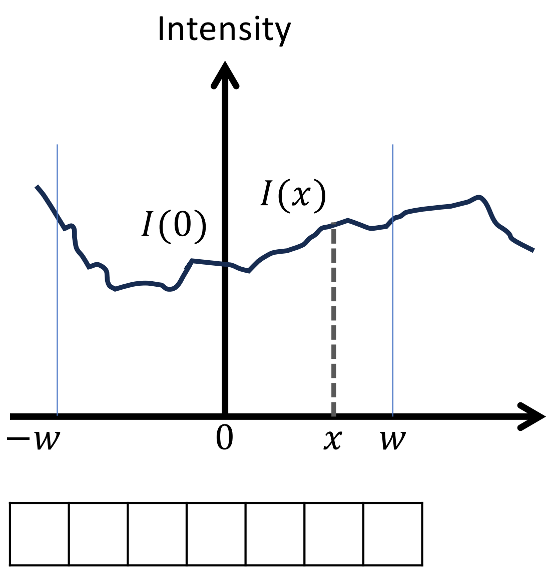
$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0) \quad + \cancel{R_n}$$

Re-write in matrix form

$$I(x) \approx \begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \ldots & \frac{1}{n!}x^n \end{bmatrix} \begin{bmatrix} I(0) \\ I'(0) \\ I''(0) \\ I'''(0) \\ \vdots \\ I^{(n)} \end{bmatrix}$$

For notational simplicity, lets refer the vector of intensity and its derivatives as $\boldsymbol{d}$

Polynomial approximation



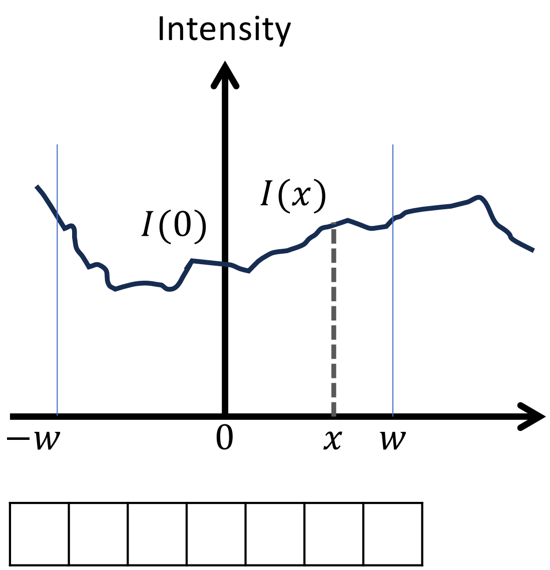
Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Example
Show the 0th order approximation

$I(x) = [1][I(0)]$

Polynomial approximation



Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Practice Question

Show the 1st and 2nd order approximations

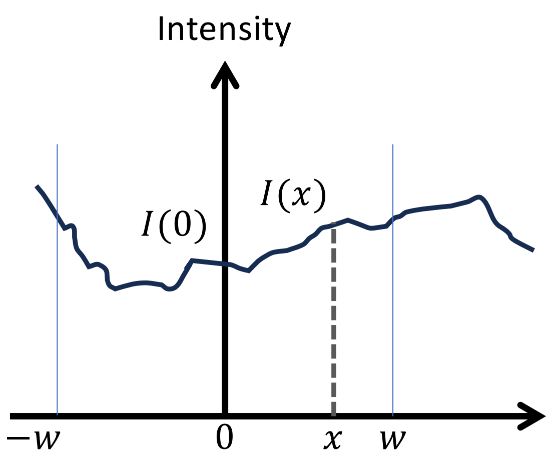
$$I(x) = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} I(0) \\ I'(0) \end{bmatrix} \quad \text{1st-order}$$

$$I(x) = \begin{bmatrix} 1 & x & x^2/2 \end{bmatrix} \begin{bmatrix} I(0) \\ I'(0) \\ I''(0) \end{bmatrix} \quad \text{2nd-order}$$

Compute derivatives at pixel 0 (i.e., the center of the pathc)

Fit a polynomial of degree n to the patch intensities

$$I(x) = I(0) + x I'(0) + \frac{x^2}{2} I''(0)$$



4	2.6	2.5	3	3.5	4	4
-3	-2	-1	0	1	2	3

derivatives at I(0)

$$I(-3) = I(0) + (-3) I'(0) + \frac{9}{2} I''(0) = 4$$

$$I(-2) = I(0) + (-2) I'(0) + 2 I''(0) = 2.6$$

$$I(-1) = I(0) + (-1) I'(0) + \frac{1}{2} I''(0) = 2.5$$

$$I(0) = I(0) = 3$$

$$I(1) = I(0) + (1) I'(0) + \frac{1}{2} I''(0) = 3.5$$

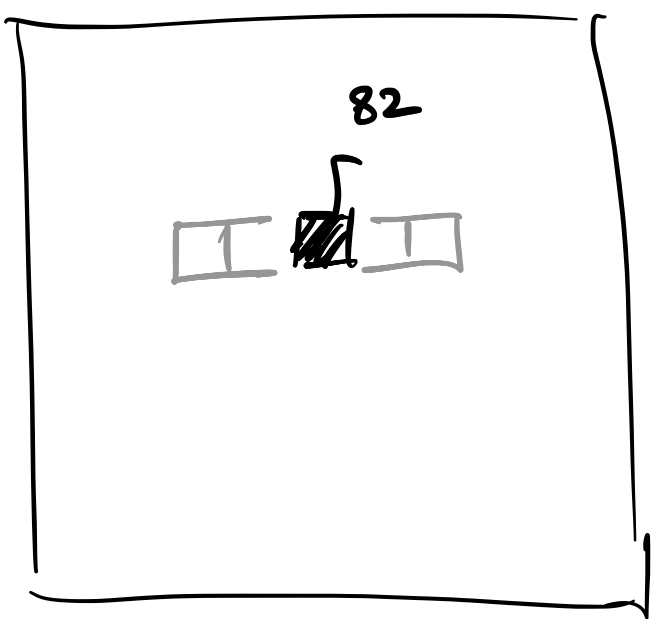
$$A\vec{x} = \vec{b}$$

?

4	2.6	2.5	3	3.5	4	4
-3	-2	-1	0	1	2	3

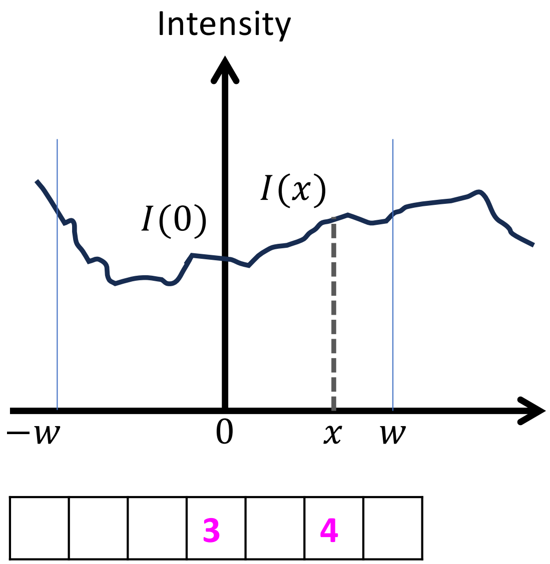
$\rightsquigarrow A\vec{x} = \vec{b}$

$$\underbrace{\begin{bmatrix} 1 & -3 & 9/2 \\ 1 & -2 & 2 \\ 1 & -1 & 1/2 \\ 1 & 0 & 0 \\ 1 & 1 & 1/2 \\ 1 & 2 & 2 \\ 1 & 3 & 9/2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I(0) \\ I'(0) \\ I''(0) \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 2.6 \\ 2.5 \\ 3 \\ 3.5 \\ 4 \\ 4 \end{bmatrix}}_{\vec{b}}$$



Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



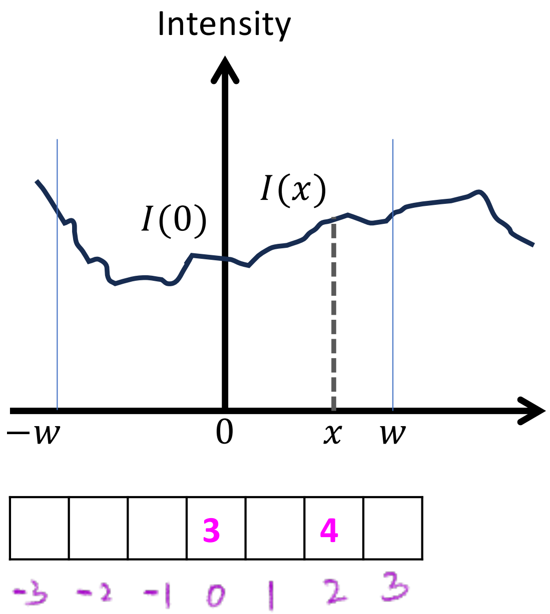
Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0)$$

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree 2

Use second-order Taylor series expansion

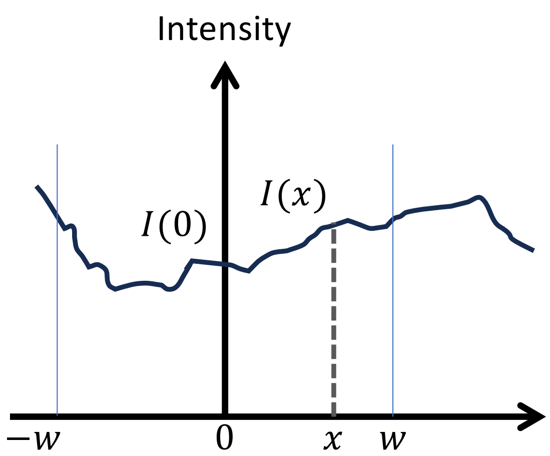
$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0)$$

Unknowns

I

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



4	2.6	2.5	3	3.5	4	4

I_1	I_2	I_3	I_4	I_5	I_6	I_7

For convenience, we refer to patch intensities as I_x where $x \in [1, 2w+1]$. Then I_{w+1} refers to the intensity at patch center.

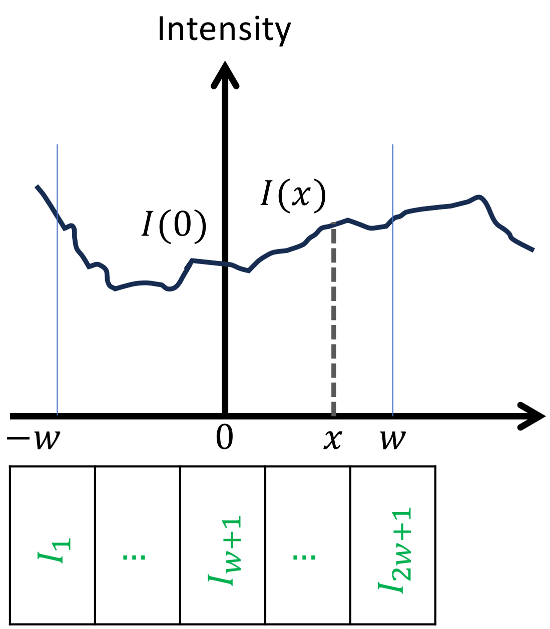
Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0)$$

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree n

Use nth order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

$(n + 1)$ Unknowns

Observation

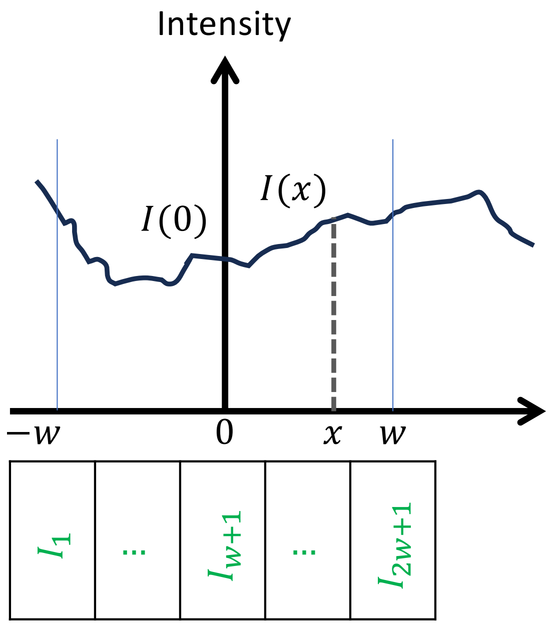
A $(2w + 1)$-patch gives $2w + 1$ equations.

Conclusion

For a patch of size $(2w + 1)$, it is only possible to fit a polynomial of degree $2w$.

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree n

Use nth order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

$(n+1)$ Unknowns

$\bar{b}$ A $\vec{x}$

$$\boldsymbol{I}_{(2w+1)\times 1} = \boldsymbol{X}_{(2w+1)\times n}\boldsymbol{d}_{n\times 1}$$

Intensities (known)

Positions (known)

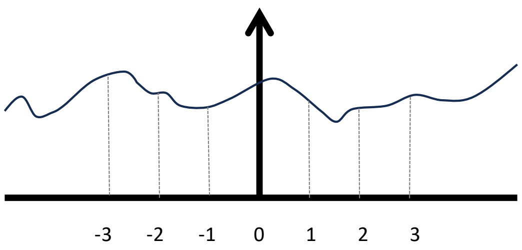
Derivatives (unknown)

Solve this linear system of equations in terms of ***d*** minimizes the fit error.

$$\boxed{\|\boldsymbol{I} - \boldsymbol{X}\boldsymbol{d}\|^2}$$

Solution ***d*** is called the *least squares fit*

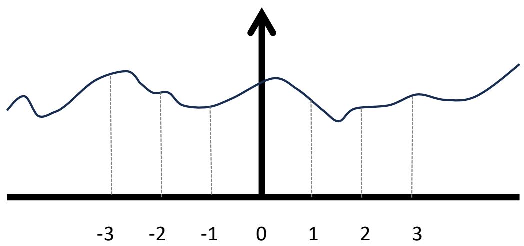
0th order estimation (constant) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [d_0]$$

0th order estimation (constant) of $I(x)$



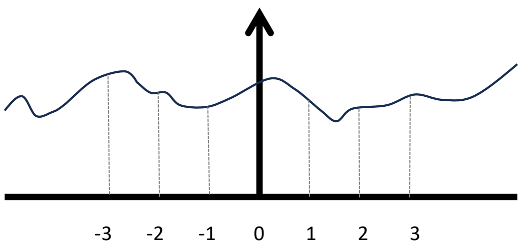
System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [d_0]$$

Solution is the mean intensity of the patch

Provides the estimate of intensity of the center of the patch

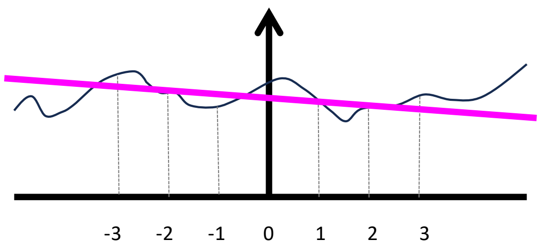
1st order estimation (linear) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$

1st order estimation (linear) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$

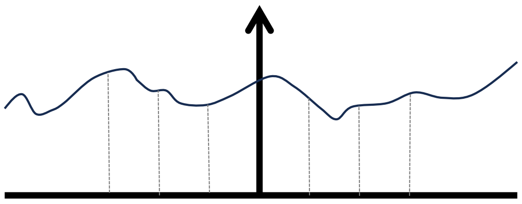
Solution minimizes the sum of vertical distance between the line and the image intensities.

Provides the estimate of intensity and its derivative at the patch center

Matrix representation of a line (in 2D)

$$y = b + mx = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix}$$

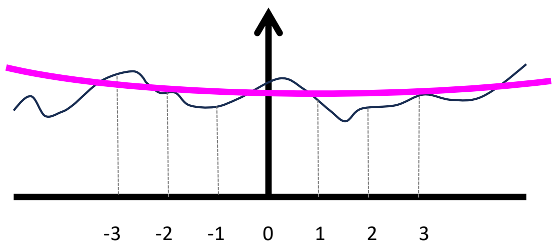
2nd order estimation (quadratic) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9/2 \\ 1 & -2 & 2 \\ 1 & -1 & 1/2 \\ 1 & 0 & 0 \\ 1 & 1 & 1/2 \\ 1 & 2 & 2 \\ 1 & 3 & 9/2 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

2nd order estimation (quadratic) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9/_2 \\ 1 & -2 & 2 \\ 1 & -1 & 1/_2 \\ 1 & 0 & 0 \\ 1 & 1 & 1/_2 \\ 1 & 2 & 2 \\ 1 & 3 & 9/_2 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

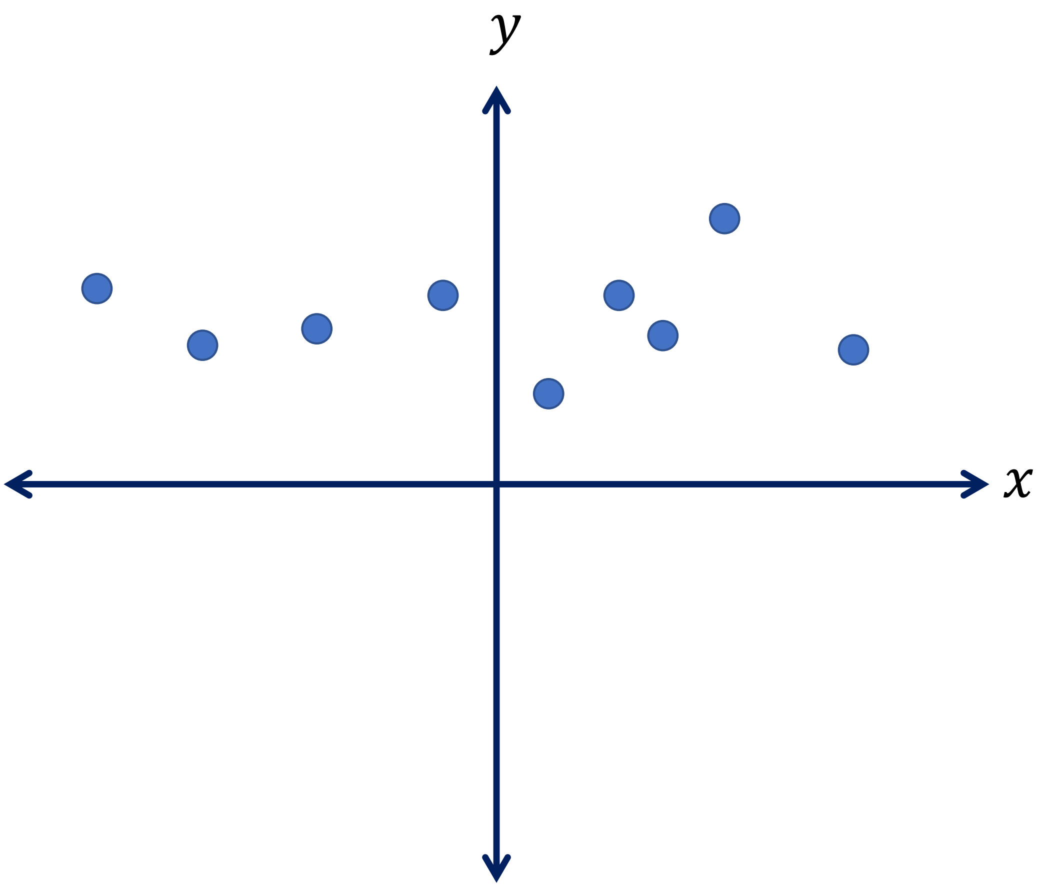
Solution fits a parabola/hyperbola/ellipse to patch intensities

Provides the estimate of intensity and its first and second derivatives at the patch center

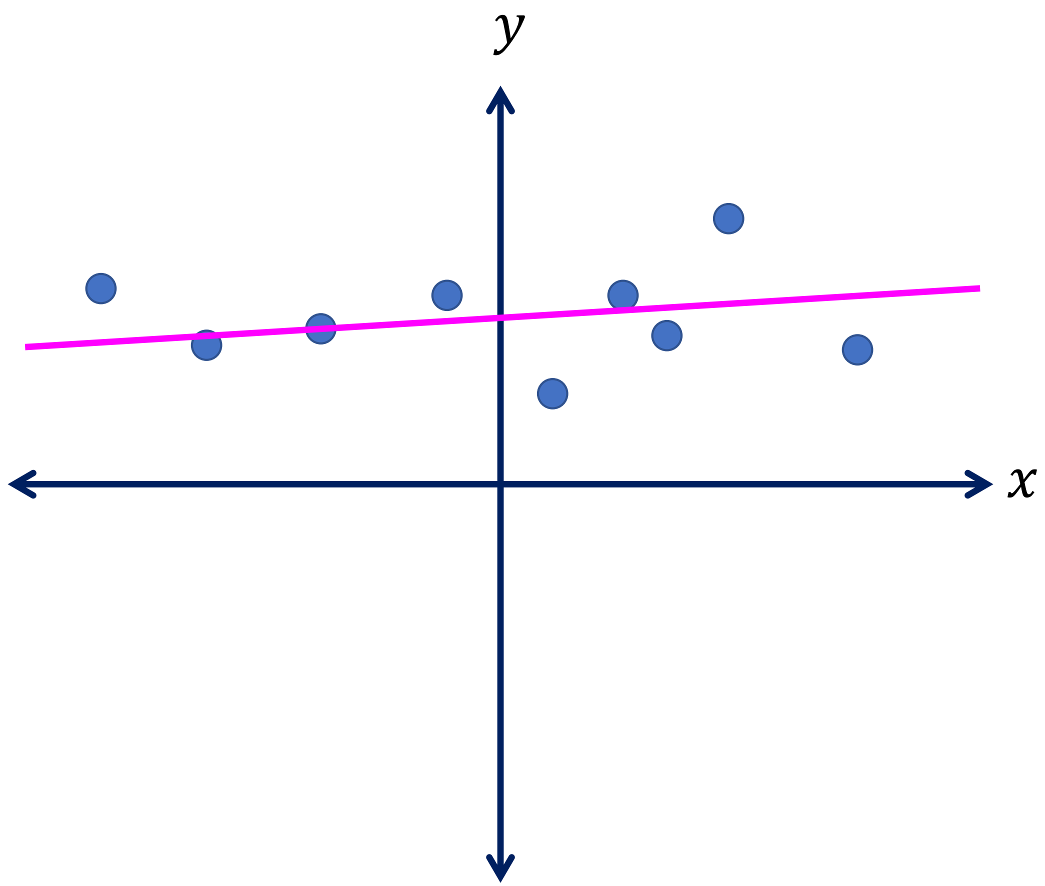
Matrix representation of second order polynomials

$$y = ax^2 + bx + c = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Least squares fitting



Least squares fitting



Least squares fitting often use the following notation to represent the system of linear equations

$$\boldsymbol{Ax} = \boldsymbol{b}$$

The solution is

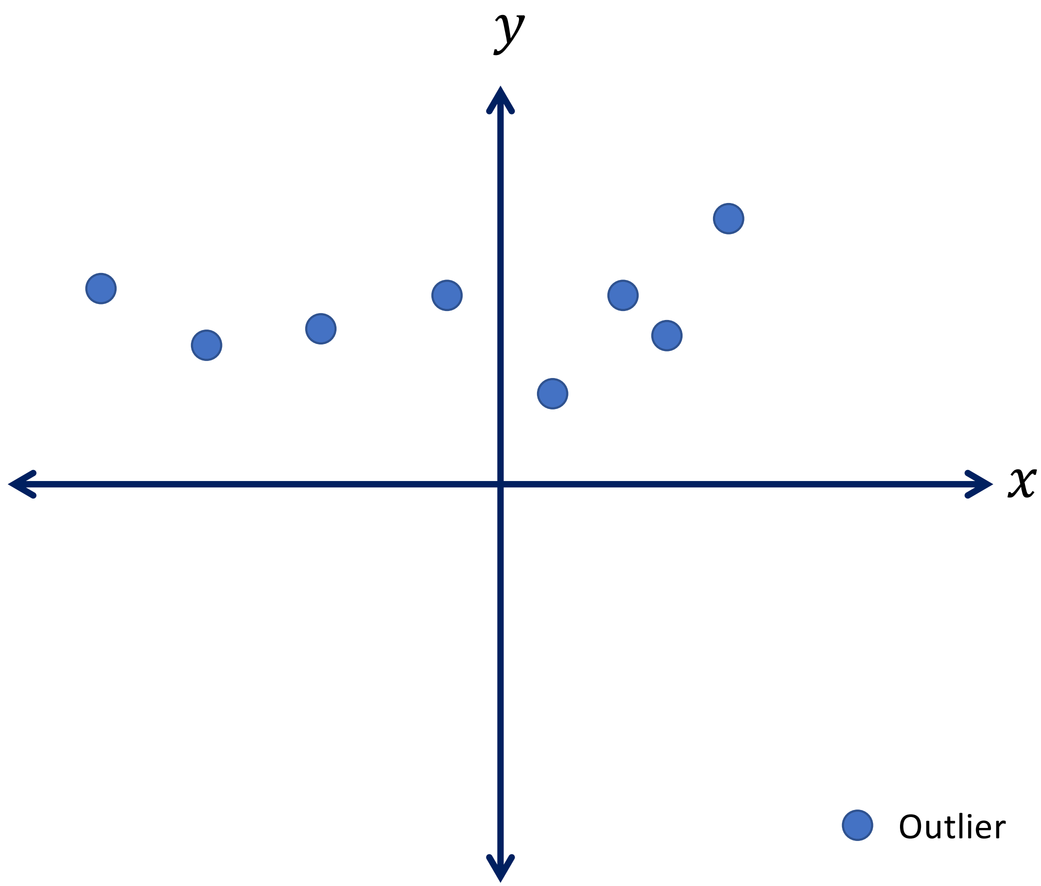
$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

where $\boldsymbol{A}^{-1}$ is inverse (or pseudo-inverse) of $\boldsymbol{A}$.

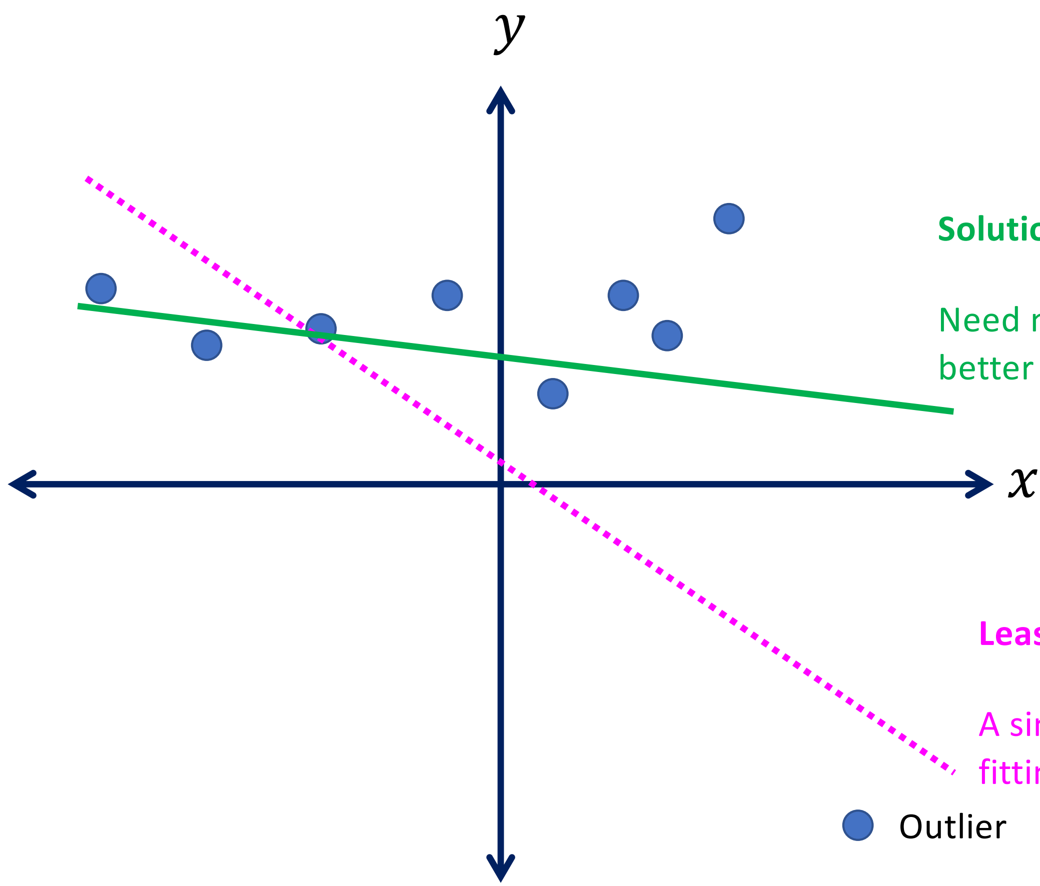
Recall that we need to solve the following system of linear equations when approximating patches with polynomials.

$$\underbrace{\boldsymbol{I}_{(2w+1)\times 1}}_{\boldsymbol{b}} = \underbrace{\boldsymbol{X}_{(2w+1)\times n}}_{\boldsymbol{A}}\underbrace{\boldsymbol{d}_{n\times 1}}_{\boldsymbol{x}}$$

Least squares fitting



Least squares fitting



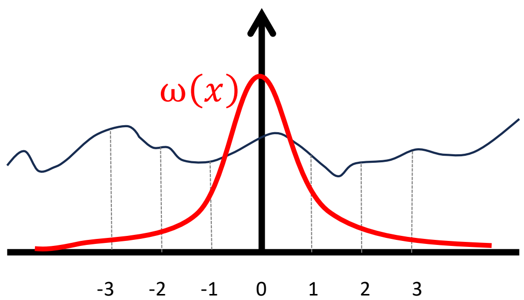
Solution: Robust least squares fitting

Need more completed methods that can better deal with outliers

Least squares fitting

A single outlier has resulted in a poor line-fitting estimate

Weighted least squares estimate of $I(x)$



Give more weight to the pixels near center and less weight to pixels that are far from center, e.g., $\omega(x) = e^{-x^2}$

Bias our estimate of $I'(0)$ towards the center of the patch.

For patch

I_1	…	I_{w+1}	…	I_{2w+1}

The system of linear equations becomes

$$\begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{2w+1} \end{bmatrix} \boldsymbol{I}_{(2w+1)\times 1} = \begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{2w+1} \end{bmatrix} \boldsymbol{X}_{(2w+1)\times n} \boldsymbol{d}_{n\times 1}$$

and the solution $\boldsymbol{d}$ minimizes the norm:

$$\left\| \begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{2w+1} \end{bmatrix} (\boldsymbol{I} - \boldsymbol{X}\boldsymbol{d}) \right\|^2$$

Estimating image derivatives

- For each row y, define a window of width $2w + 1$ at pixel (i.e., column) x
  - Fit a polynomial (usually of degree 1 or 2)
  - Assign the fitted polynomial's derivates at location 0 (i.e., center of the patch, or column y in the image space)
  - Slide the window one over, until the end of the row

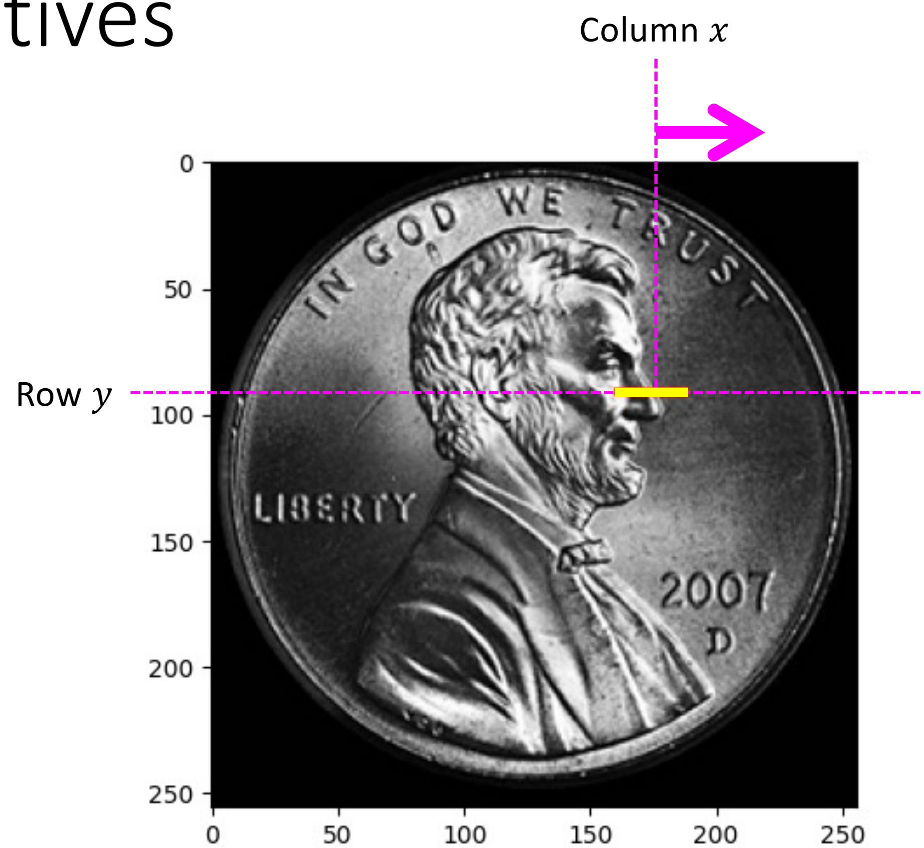
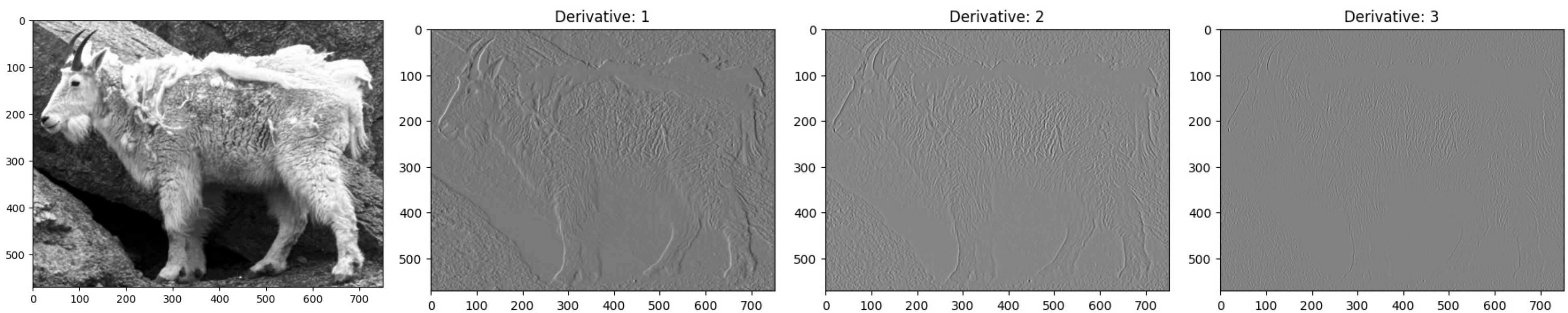


Image derivatives

Fitting a 3rd-order Taylor series using a 5-pixel patch



3   2   0   4   3   5

0   1   2   3   4   5

-1   0   1   2

0.2   2.2

$$ax^3 + bx^2 + cx + d = f(x)$$

$$\underbrace{\begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 4 \\ 3 \end{bmatrix}}_{\vec{b}}$$

$3ax^2 + 2bx + c$ 1st deriv.

Summary

- 1D image patches
- Approximating 1D image patches via polynomials
- Computing image derivatives via fitting polynomials
- Least squares solution to a system of linear equations
- Weighted least squares