Polynomial Approximation

Computational Photography (CSCI 3240U)

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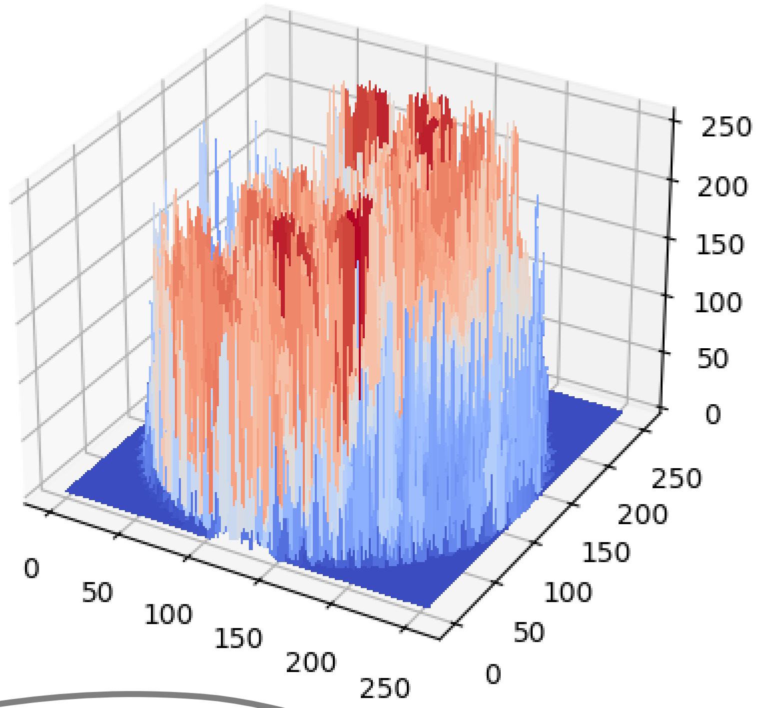
Today's lecture

- How to compute image derivatives by fitting polynomials to 1D image patches?
  - Taylor series expansion around a patch center
  - Least square fitting of a system of linear equations

Image as a surface in 3D

Consider a gray-scale image $I(x, y)$ then the height of the surface at (x, y) is $I(x, y)$. The surface passes through the 3D point $(x, y, I(x, y))$.



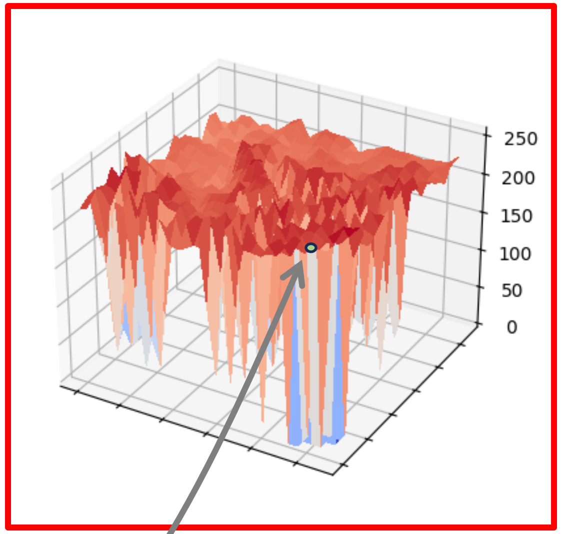
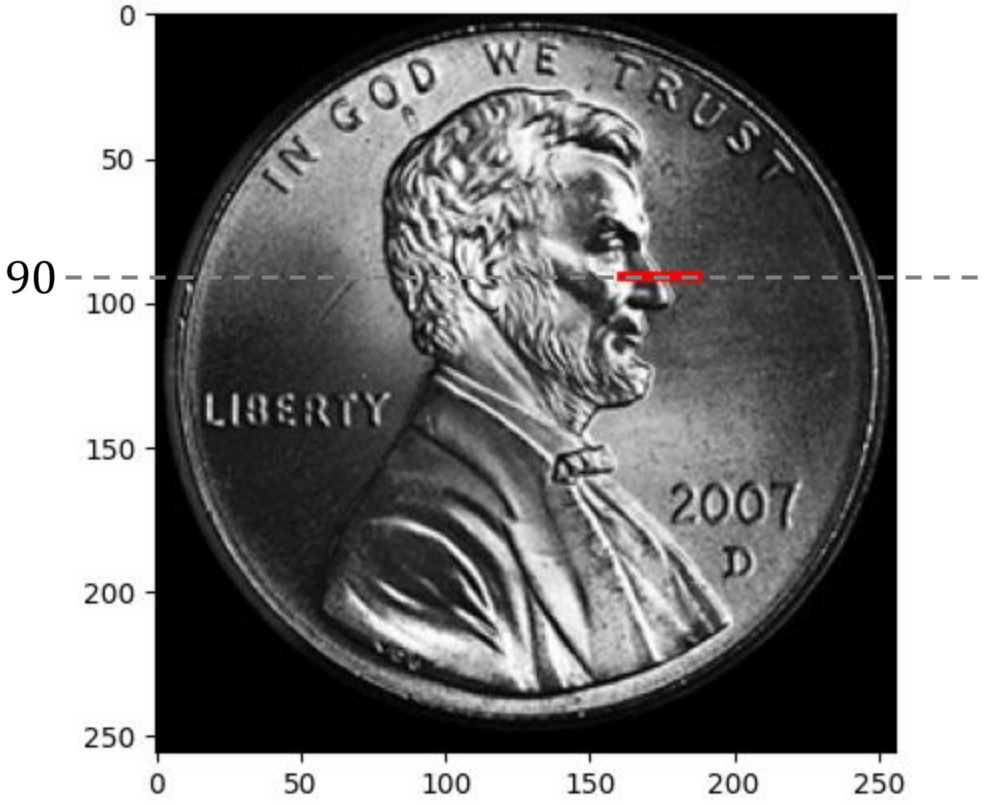
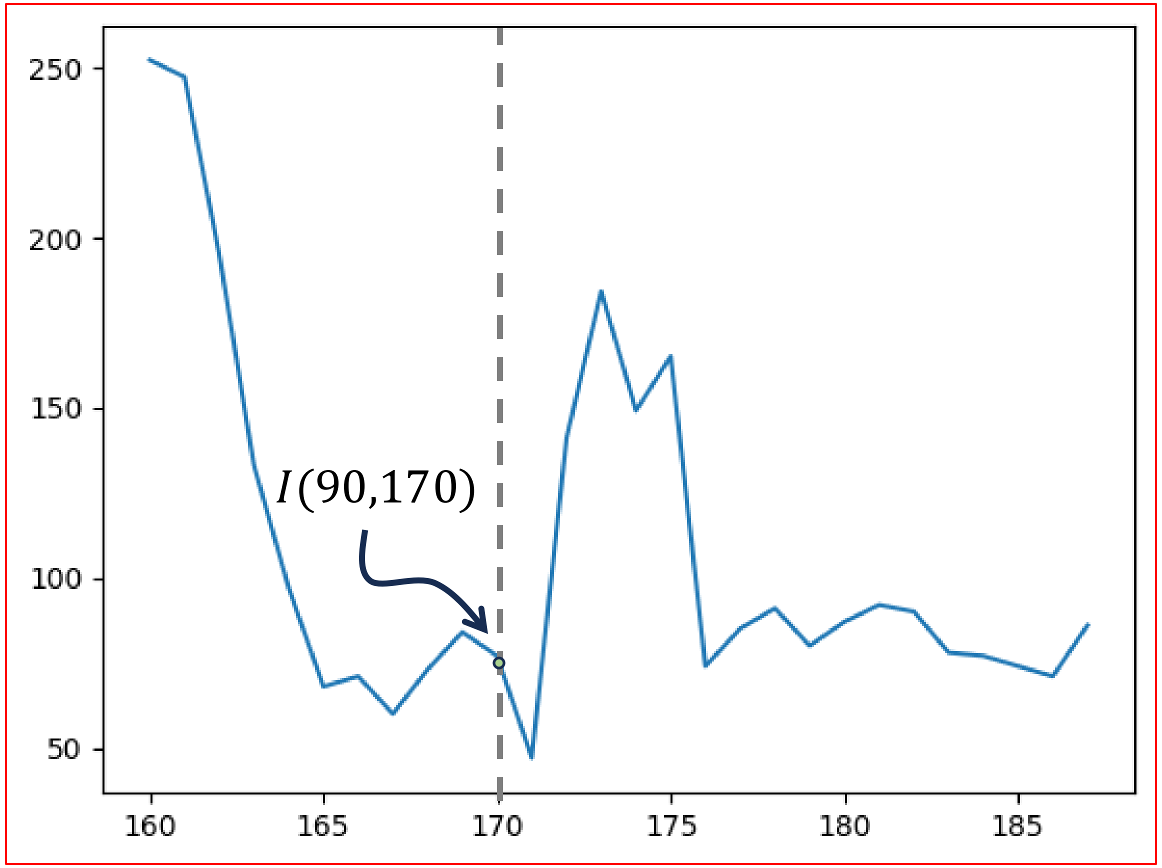


Image rows (or columns) as 2D graphs





Paths as curves in 2D

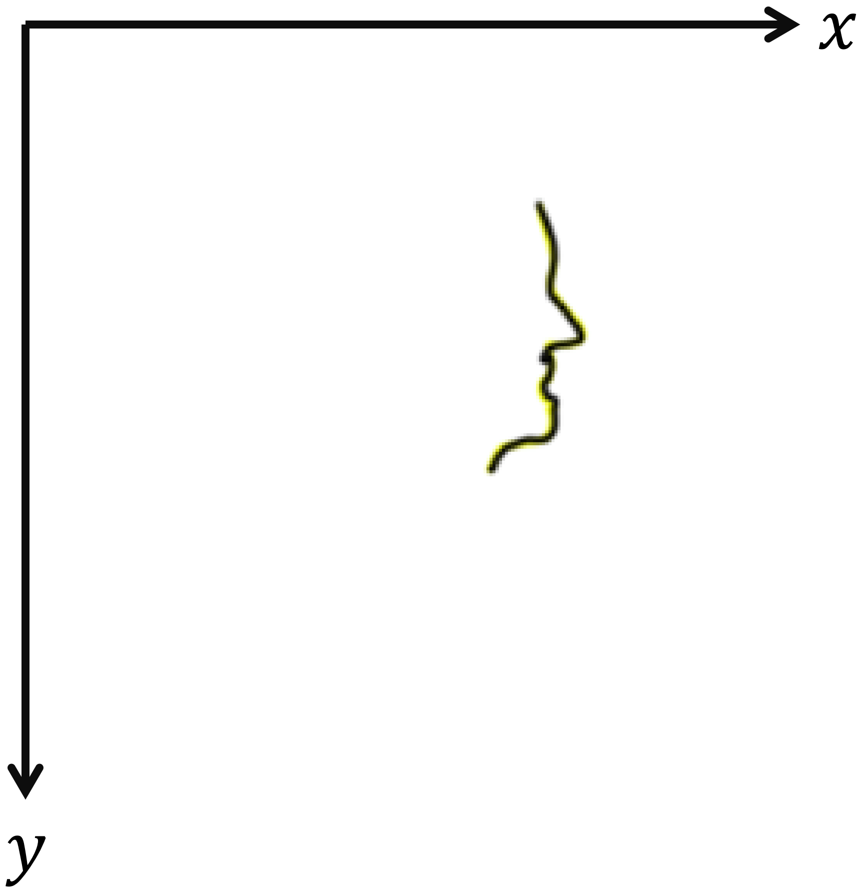
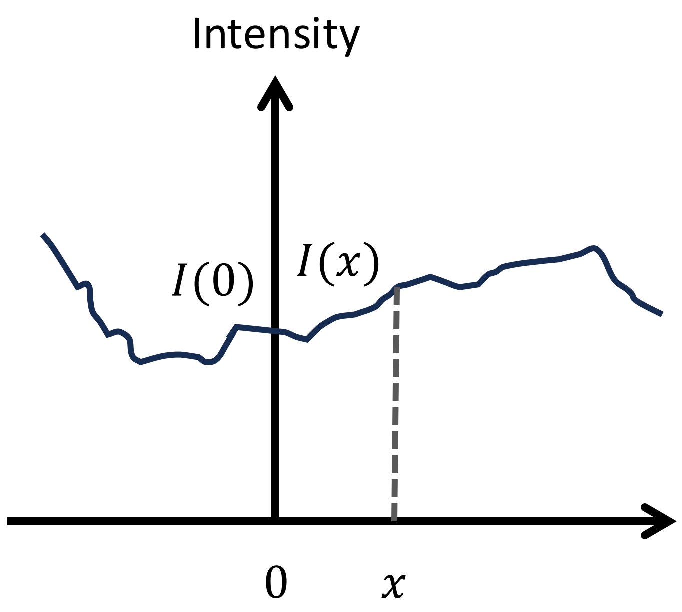


Image rows (or columns) as 2D graphs

Polynomial approximation

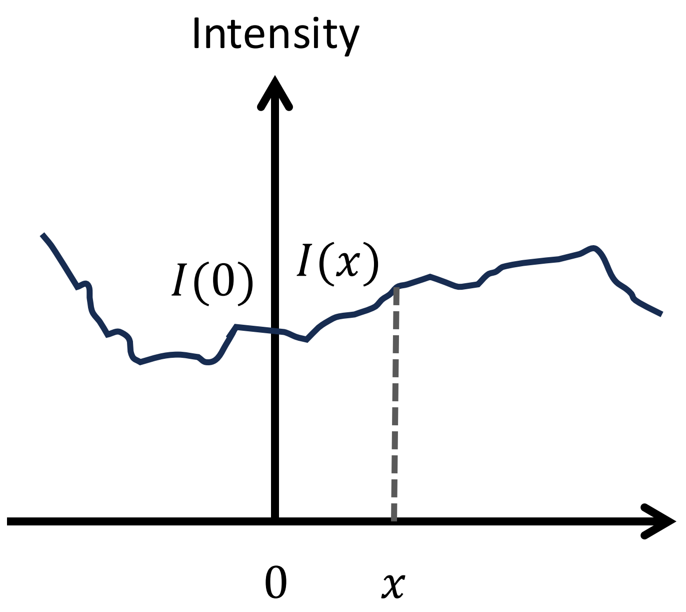


Taylor series expansion of $I(x)$ near the "patch" center 0

$I(x) = ?$

Image rows (or columns) as 2D graphs

Polynomial approximation

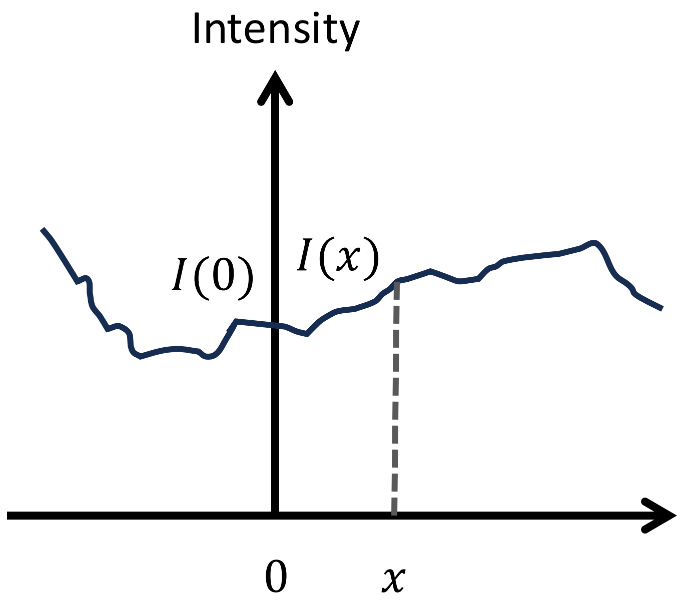


Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = I(0)$$

Image rows (or columns) as 2D graphs

Polynomial approximation

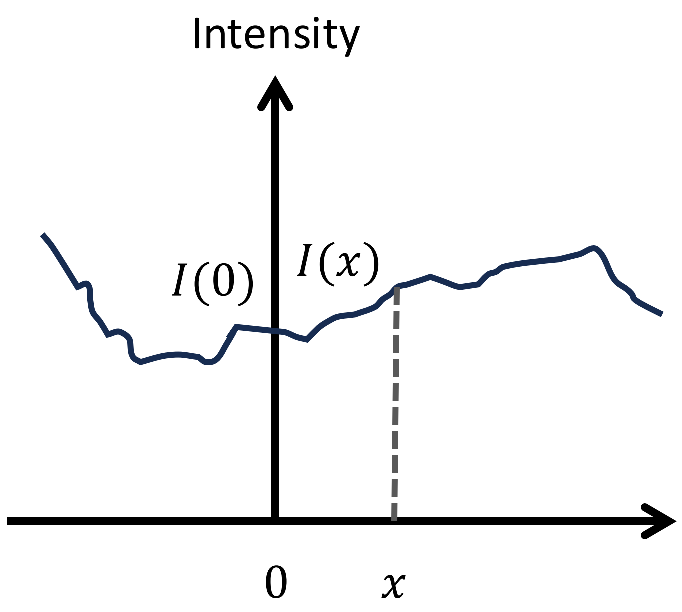


Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = I(0) + xI'(0)$$

Image rows (or columns) as 2D graphs

Polynomial approximation

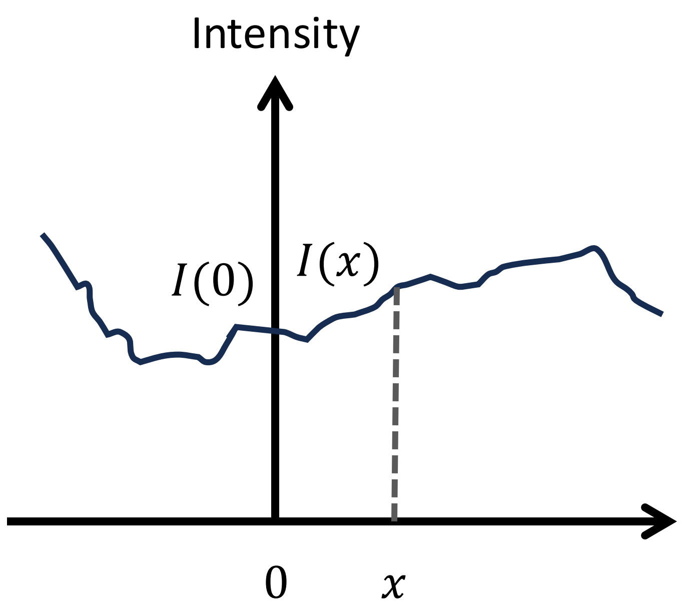


Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = I(0) + xI'(0) + \frac{x^2}{2!} I''(0)$$

Image rows (or columns) as 2D graphs

Polynomial approximation



Taylor series expansion of $I(x)$ near the "patch" center 0

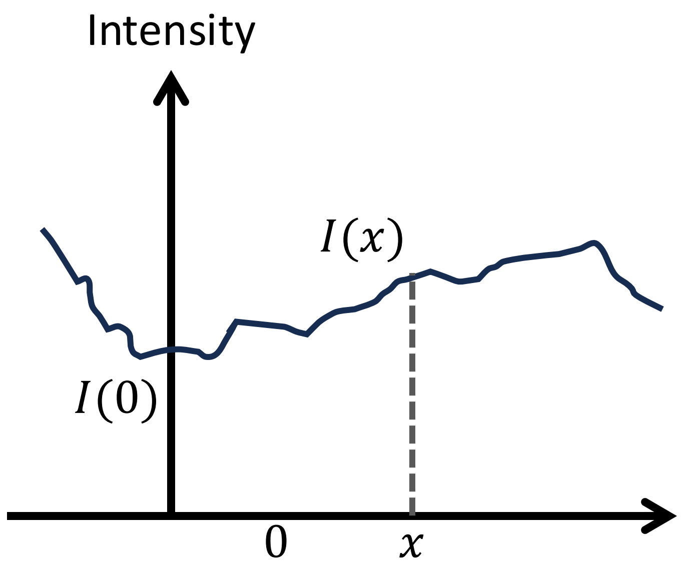
$$I(x) = I(0) + xI'(0) + \frac{x^2}{2!}I''(0) + \frac{x^3}{3!}I'''(0) + \cdots + \frac{x^n}{n!}I^{(n)} + R_{n+1}(x)$$

The residual $R_{n+1}(x)$ satisfies:

$$\lim_{x \to 0} R_{n+1}(x) = 0$$

Image rows (or columns) as 2D graphs

Polynomial approximation



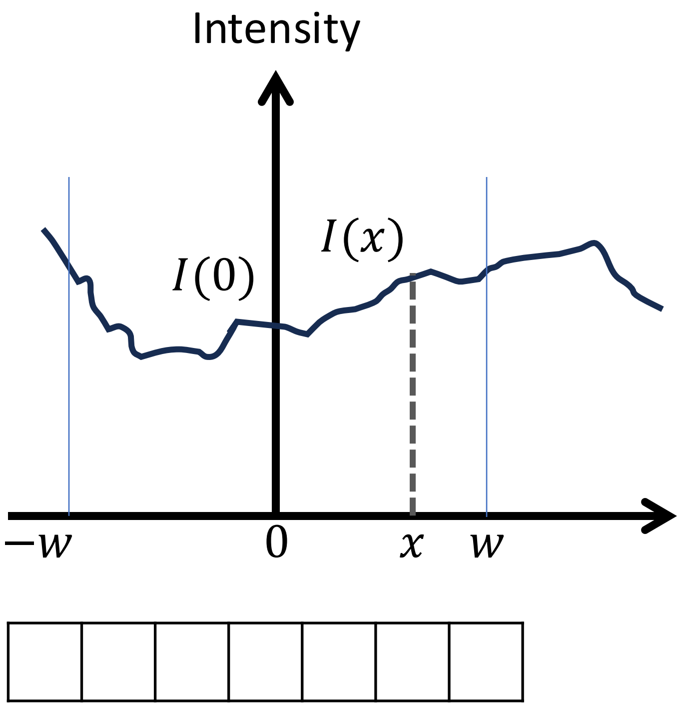
Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = I(0) + xI'(0) + \frac{x^2}{2!}I''(0) + \frac{x^3}{3!}I'''(0) + \cdots + \frac{x^n}{n!}I^{(n)} + R_{n+1}(x)$$

Nth order approximation

For a given x, approximation depends on $(n + 1)$ constants corresponding to the intensity derivative at the patch origin.

Polynomial approximation

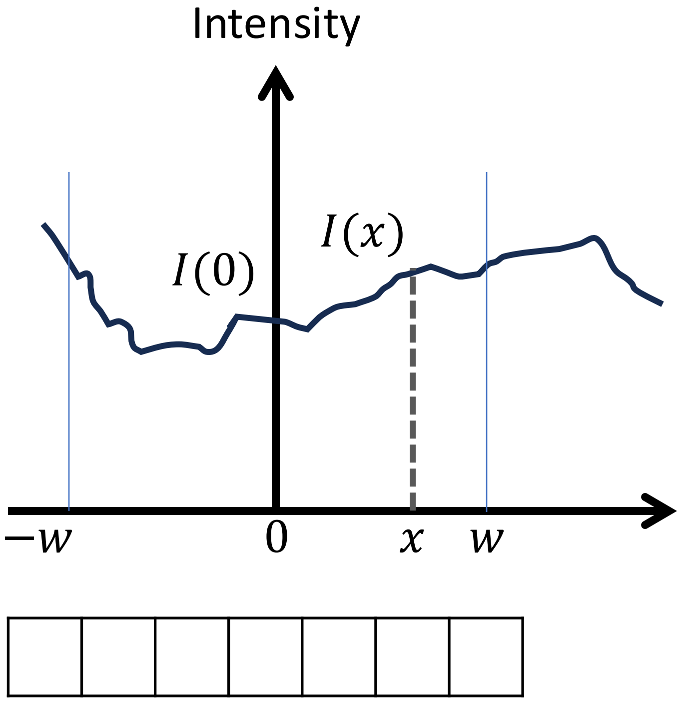


Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Re-write in matrix form

Polynomial approximation



Taylor series expansion of $I(x)$ near the patch center 0

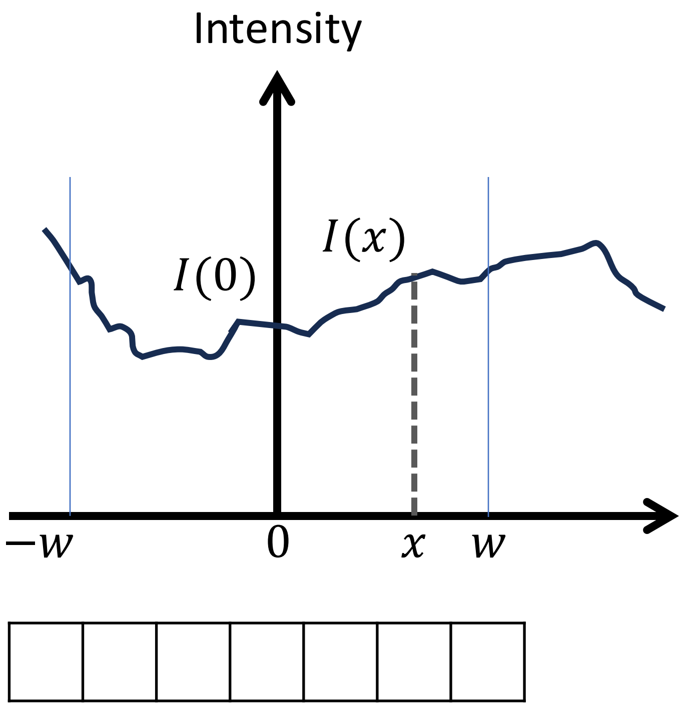
$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Re-write in matrix form

$$I(x) \approx \begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \ldots & \frac{1}{n!}x^n \end{bmatrix} \begin{bmatrix} I(0) \\ I'(0) \\ I''(0) \\ I'''(0) \\ \vdots \\ I^{(n)} \end{bmatrix}$$

For notational simplicity, lets refer the vector of intensity and its derivatives as $\boldsymbol{d}$

Polynomial approximation



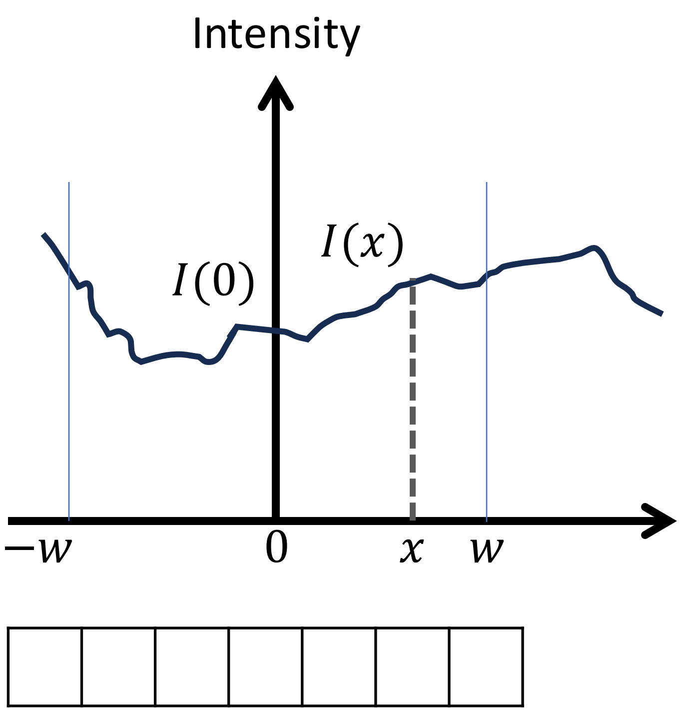
Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Example

Show the 0^{th} order approximation

Polynomial approximation



Taylor series expansion of $I(x)$ near the patch center 0

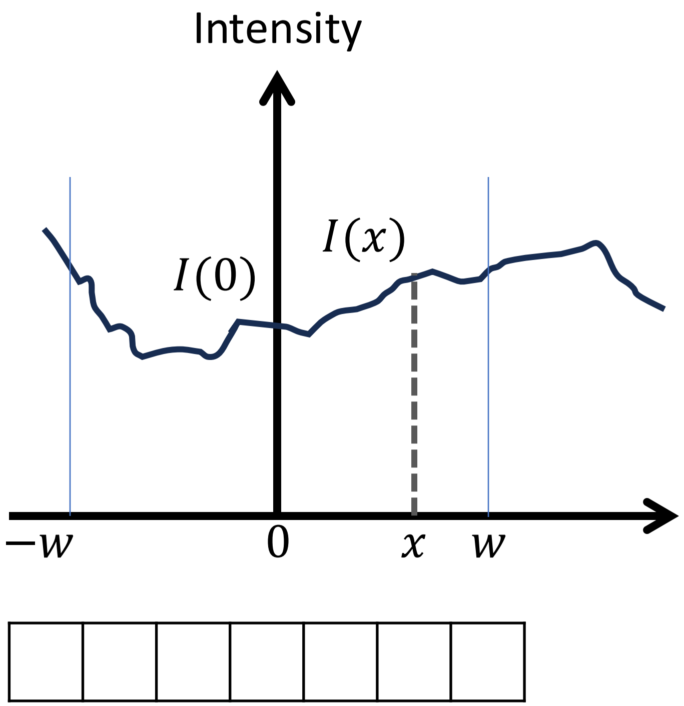
$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Practice Question

Show the 1st and 2nd order approximations

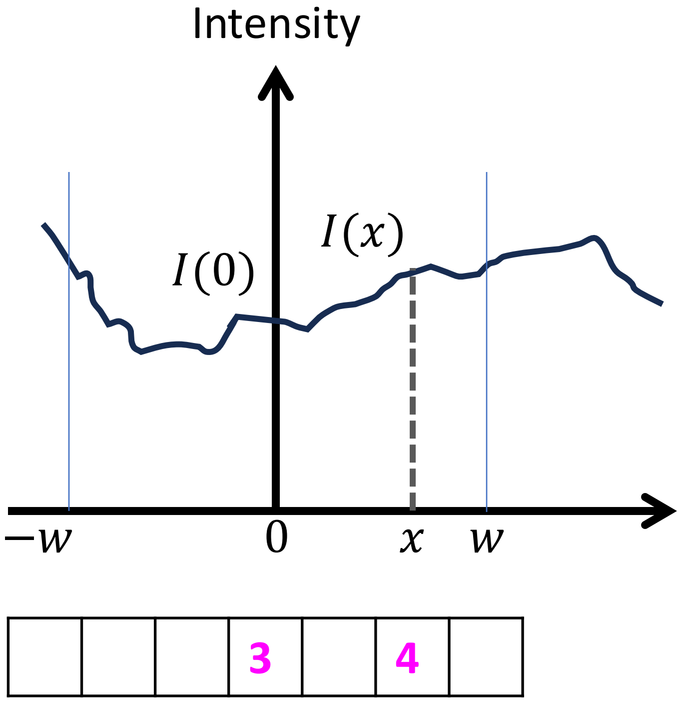
Compute derivatives at pixel 0 (i.e., the center of the pathc)

Fit a polynomial of degree n to the patch intensities



Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



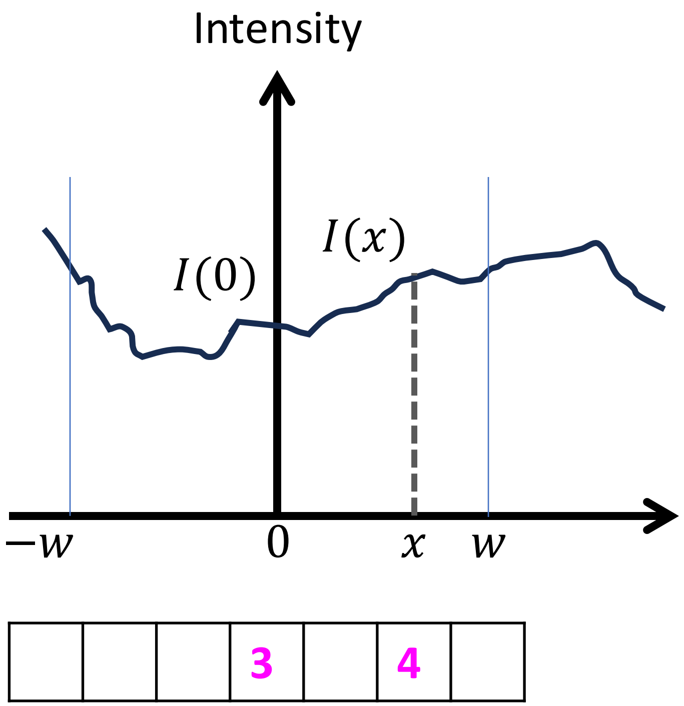
Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0)$$

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree 2

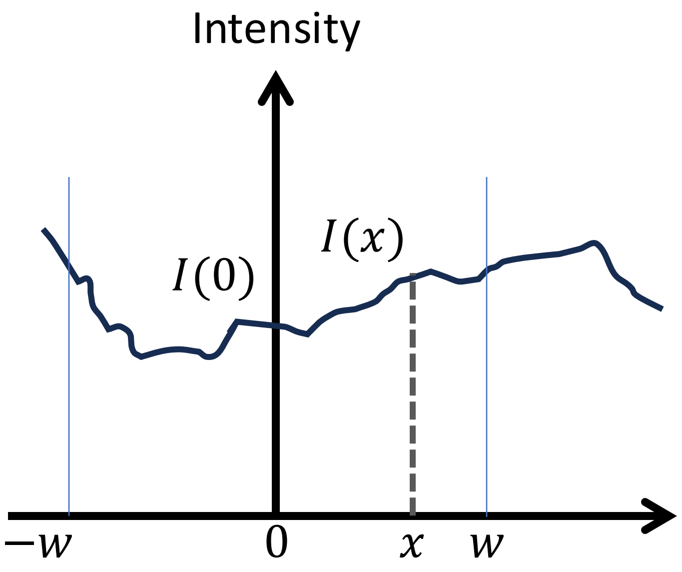
Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0)$$

Unknowns

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



4	2.6	2.5	3	3.5	4	4

I_1	I_2	I_3	I_4	I_5	I_6	I_7

For convenience, we refer to patch intensities as I_x where $x \in [1, 2w+1]$. Then I_{w+1} refers to the intensity at patch center.

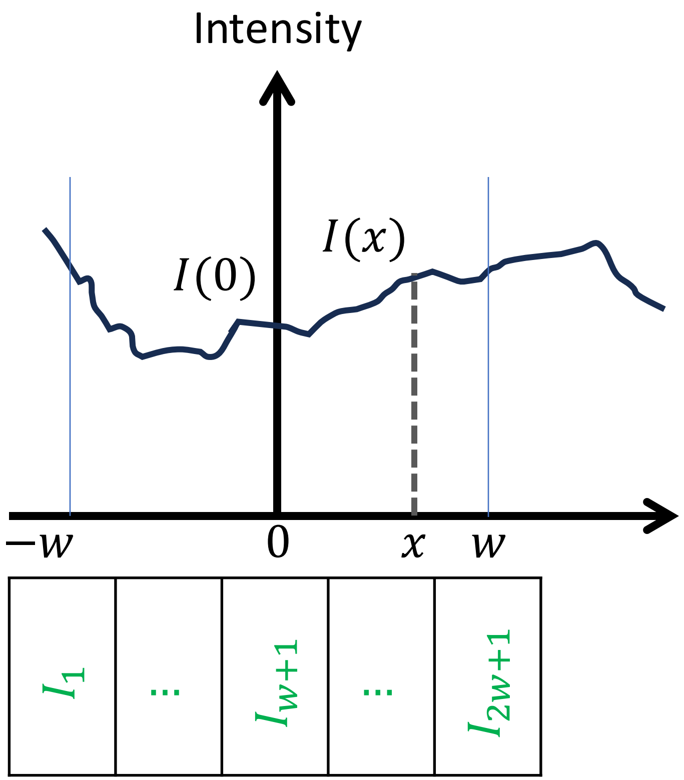
Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0)$$

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree n

Use nth order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

$(n+1)$ Unknowns

Observation

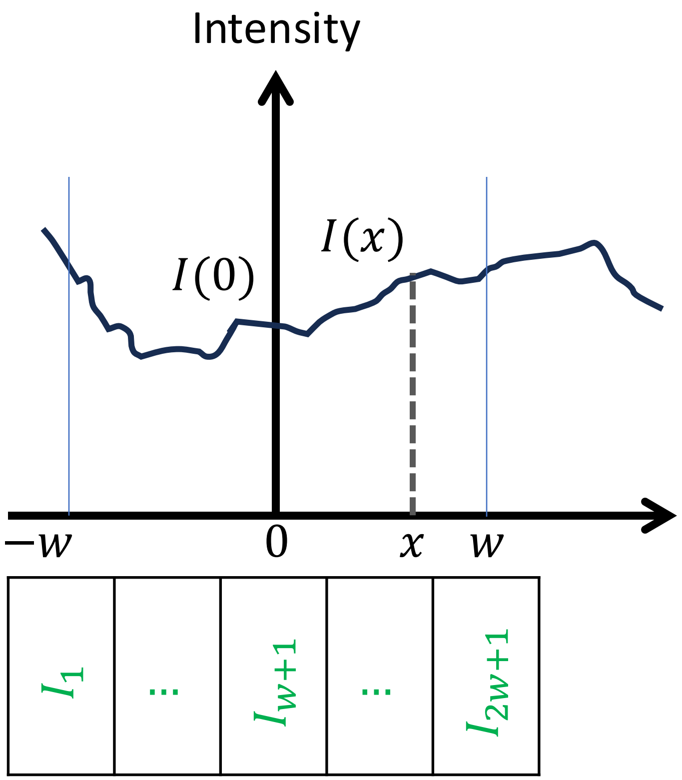
A $(2w+1)$-patch gives $2w+1$ equations.

Conclusion

For a patch of size $(2w+1)$, it is only possible to fit a polynomial of degree $2w$.

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree n

Use nth order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

$(n+1)$ Unknowns

$$\boldsymbol{I}_{(2w+1)\times 1} = \boldsymbol{X}_{(2w+1)\times n}\boldsymbol{d}_{n\times 1}$$

Intensities (known)

Positions (known)

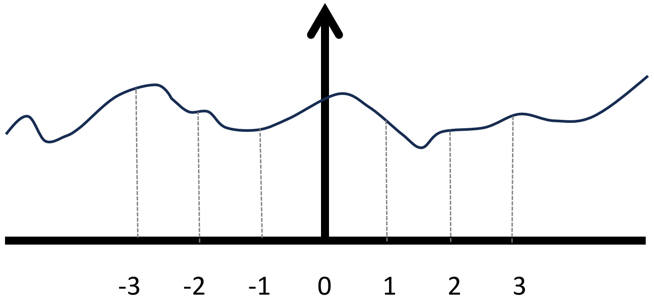
Derivatives (unknown)

Solve this linear system of equations in terms of ***d*** minimizes the fit error.

$$\|\boldsymbol{I} - \boldsymbol{X}\boldsymbol{d}\|^2$$

Solution ***d*** is called the *least squares fit*

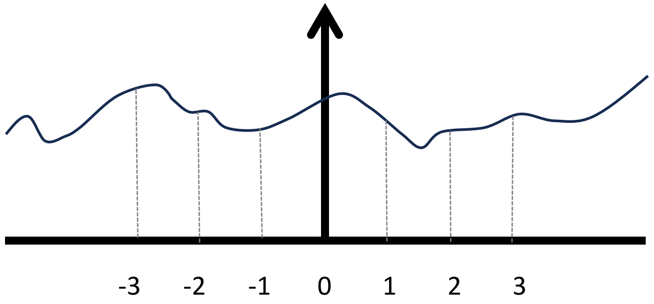
0th order estimation (constant) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [d_0]$$

0th order estimation (constant) of $I(x)$



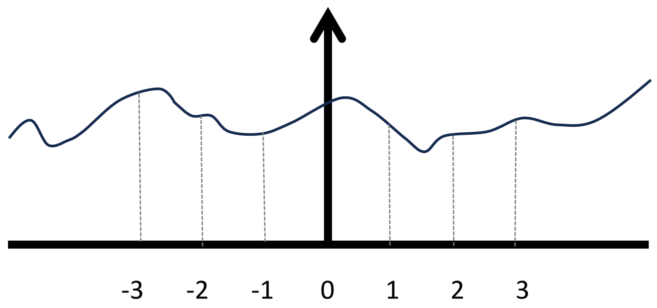
System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [d_0]$$

Solution is the mean intensity of the patch

Provides the estimate of intensity of the center of the patch

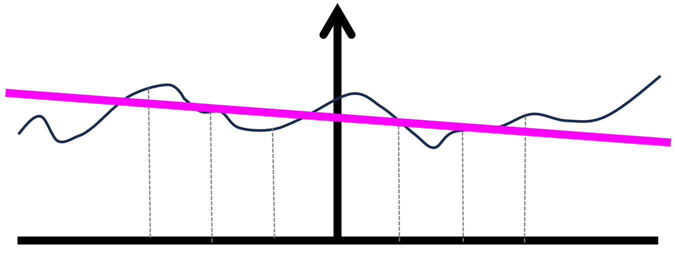
1st order estimation (linear) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$

1st order estimation (linear) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$

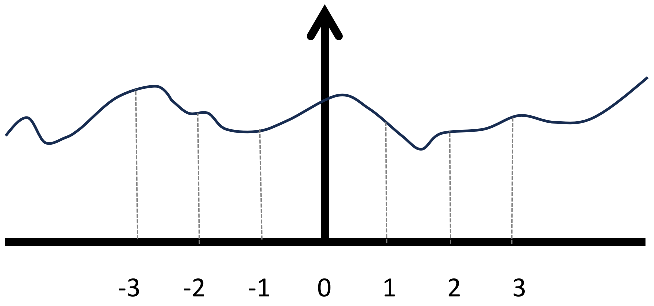
Solution minimizes the sum of vertical distance between the line and the image intensities.

Provides the estimate of intensity and its derivative at the patch center

Matrix representation of a line (in 2D)

$$y = b + mx = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix}$$

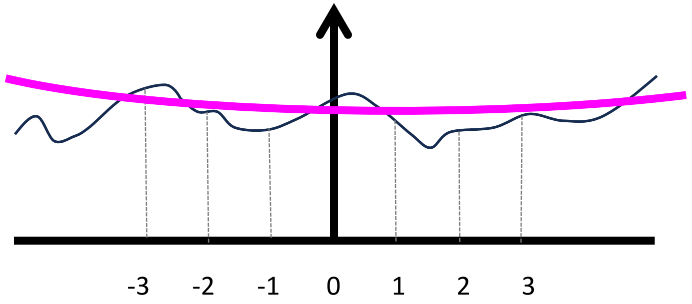
2nd order estimation (quadratic) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & {}^{9}/_{2} \\ 1 & -2 & 2 \\ 1 & -1 & {}^{1}/_{2} \\ 1 & 0 & 0 \\ 1 & 1 & {}^{1}/_{2} \\ 1 & 2 & 2 \\ 1 & 3 & {}^{9}/_{2} \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

2nd order estimation (quadratic) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9/2 \\ 1 & -2 & 2 \\ 1 & -1 & 1/2 \\ 1 & 0 & 0 \\ 1 & 1 & 1/2 \\ 1 & 2 & 2 \\ 1 & 3 & 9/2 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

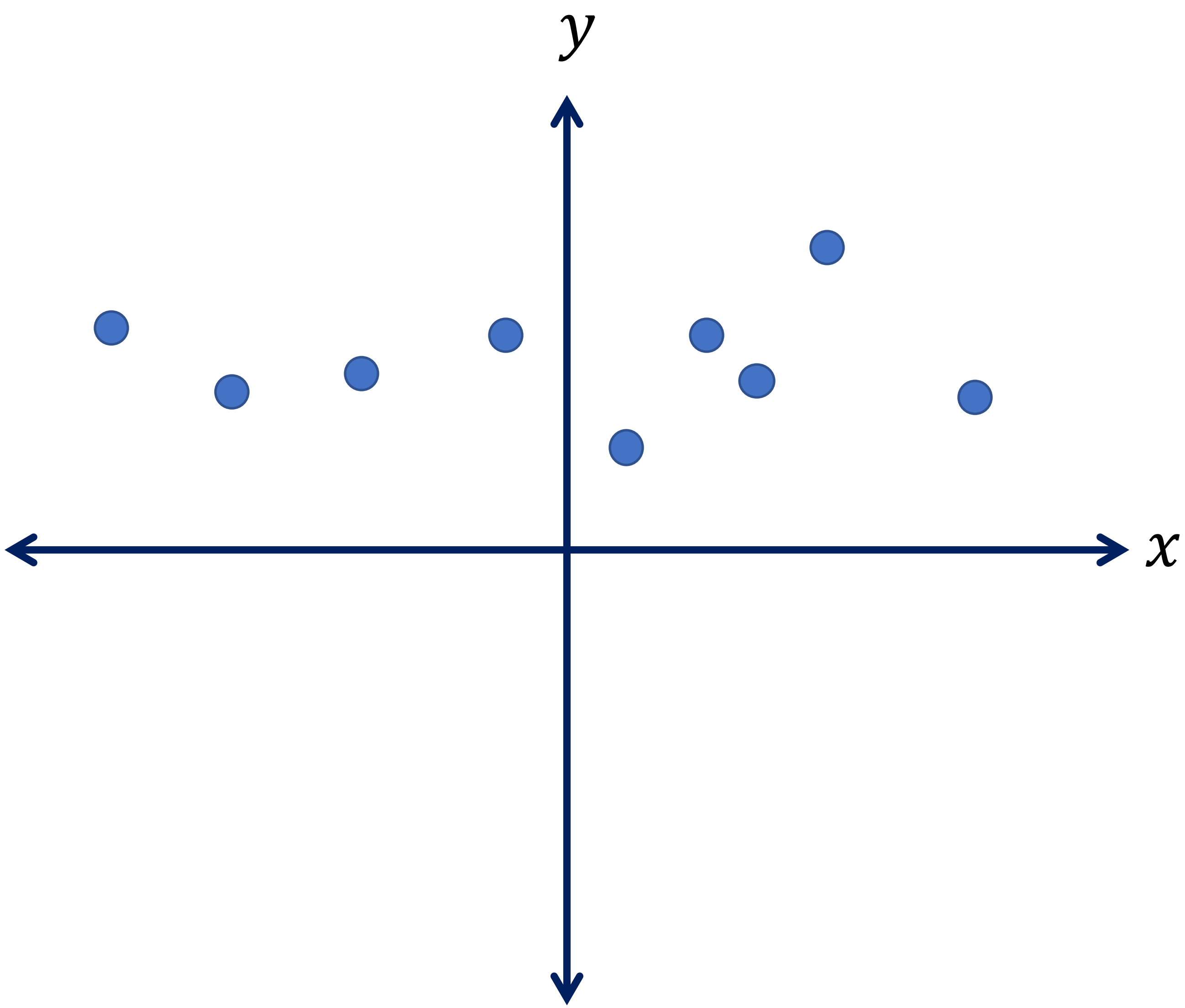
Solution fits a parabola/hyperbola/ellipse to patch intensities

Provides the estimate of intensity and its first and second derivatives at the patch center

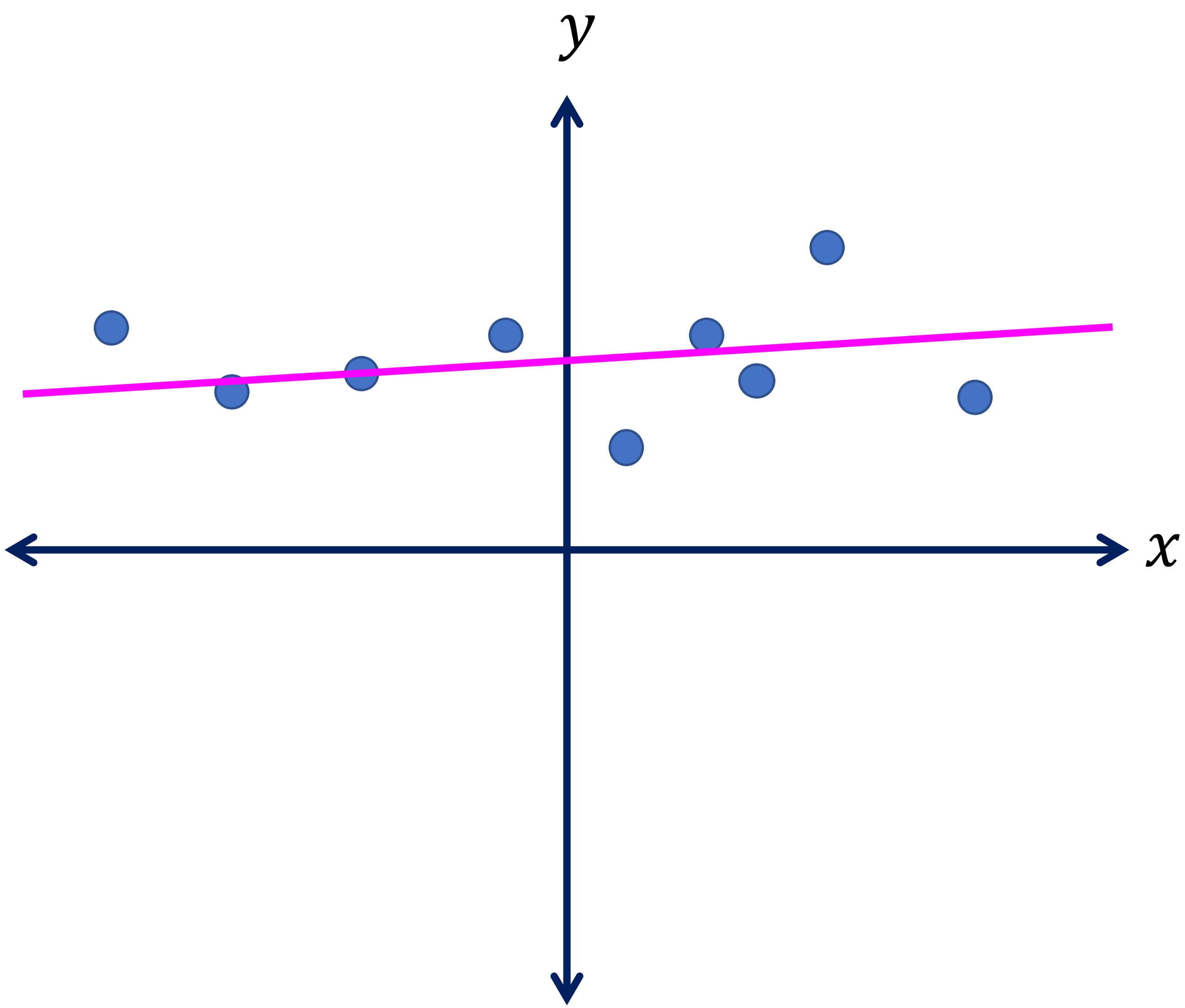
Matrix representation of second order polynomials

$$y = ax^2 + bx + c = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Least squares fitting



Least squares fitting



Least squares fitting often use the following notation to represent the system of linear equations

$$\boldsymbol{Ax} = \boldsymbol{b}$$

The solution is

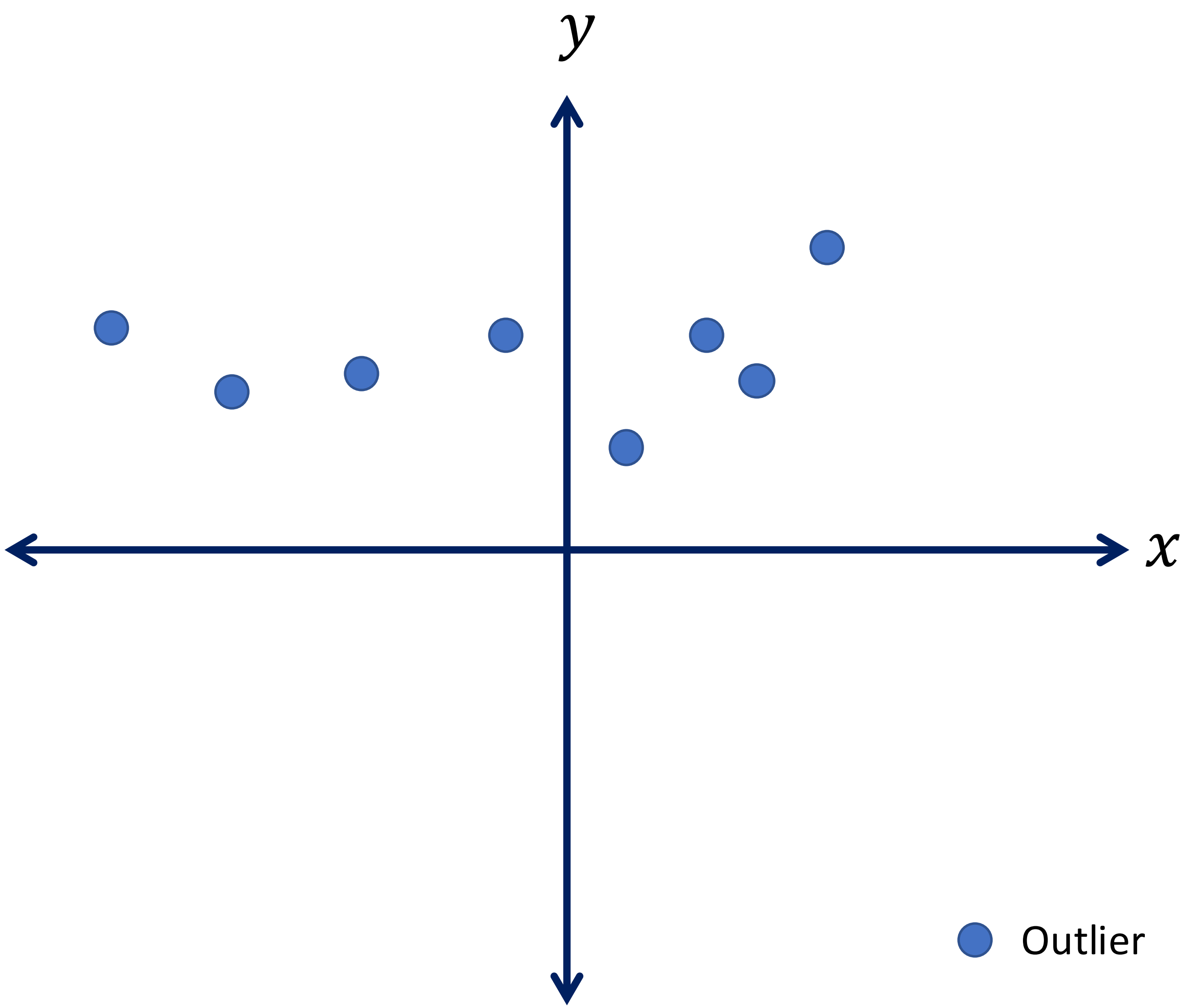
$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

where $\boldsymbol{A}^{-1}$ is inverse (or pseudo-inverse) of $\boldsymbol{A}$.

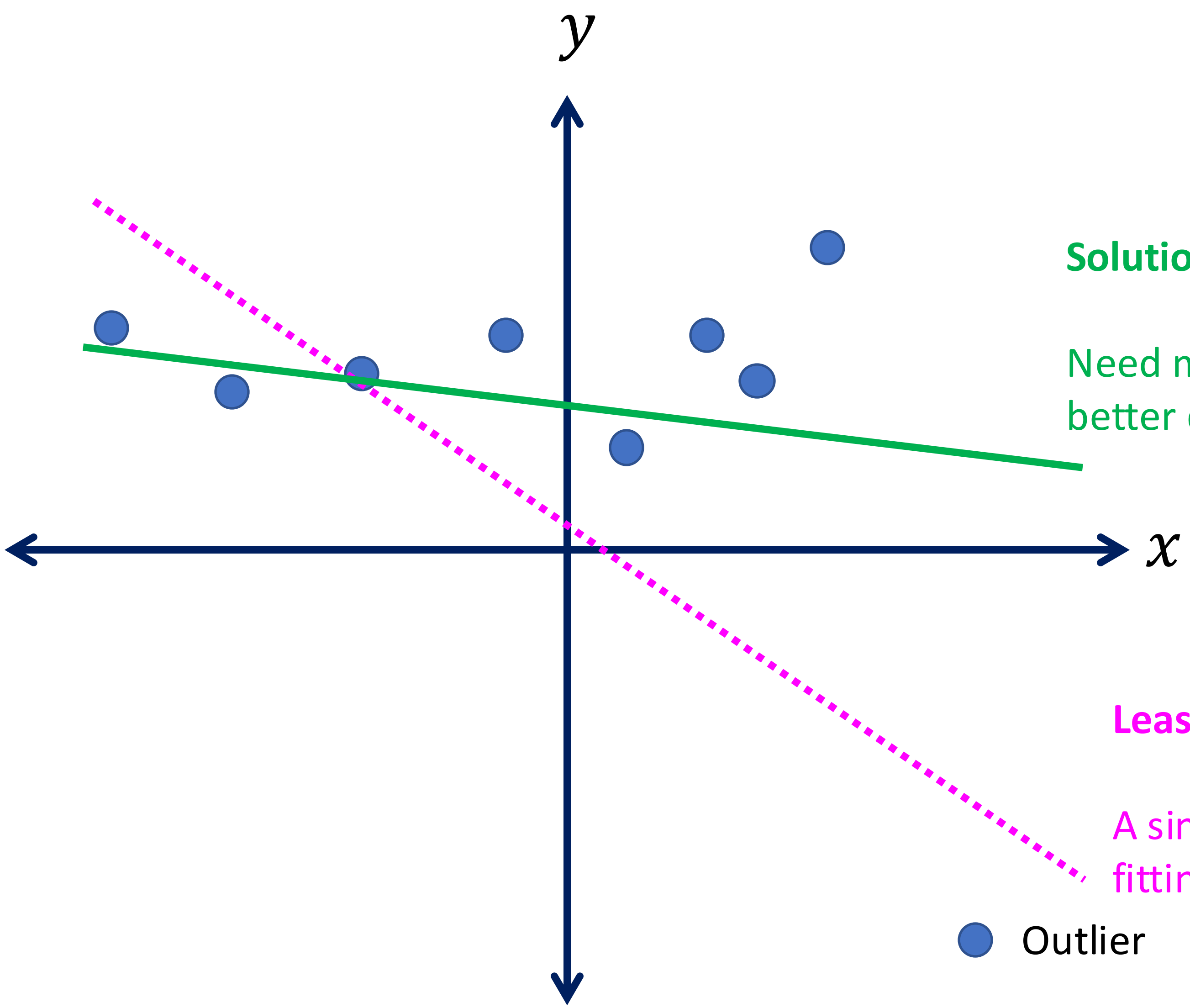
Recall that we need to solve the following system of linear equations when approximating patches with polynomials.

$$\underbrace{\boldsymbol{I}_{(2w+1)\times 1}}_{\boldsymbol{b}} = \underbrace{\boldsymbol{X}_{(2w+1)\times n}}_{\boldsymbol{A}} \underbrace{\boldsymbol{d}_{n\times 1}}_{\boldsymbol{x}}$$

Least squares fitting



Least squares fitting



Solution: Robust least squares fitting

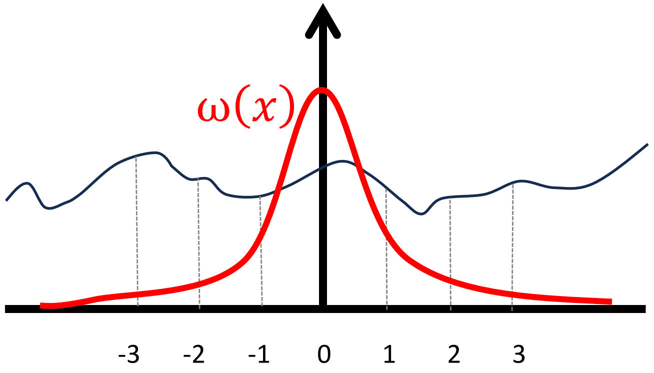
Need more completed methods that can better deal with outliers

Least squares fitting

A single outlier has resulted in a poor line-fitting estimate

Outlier

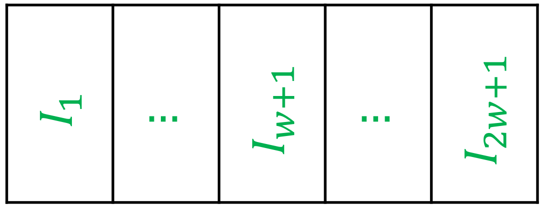
Weighted least squares estimate of $I(x)$



Give more weight to the pixels near center and less weight to pixels that are far from center, e.g., $\omega(x) = e^{-x^2}$

Bias our estimate of $I'(0)$ towards the center of the patch.

For patch



The system of linear equations becomes

$$\begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{2w+1} \end{bmatrix} \boldsymbol{I}_{(2w+1)\times 1} = \begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{2w+1} \end{bmatrix} \boldsymbol{X}_{(2w+1)\times n} \boldsymbol{d}_{n\times 1}$$

and the solution $\boldsymbol{d}$ minimizes the norm:

$$\left\| \begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{2w+1} \end{bmatrix} (\boldsymbol{I} - \boldsymbol{X}\boldsymbol{d}) \right\|^2$$

Estimating image derivatives

- For each row y, define a window of width $2w + 1$ at pixel (i.e., column) x
  - Fit a polynomial (usually of degree 1 or 2)
  - Assign the fitted polynomial's derivates at location 0 (i.e., center of the patch, or column y in the image space)
  - Slide the window one over, until the end of the row

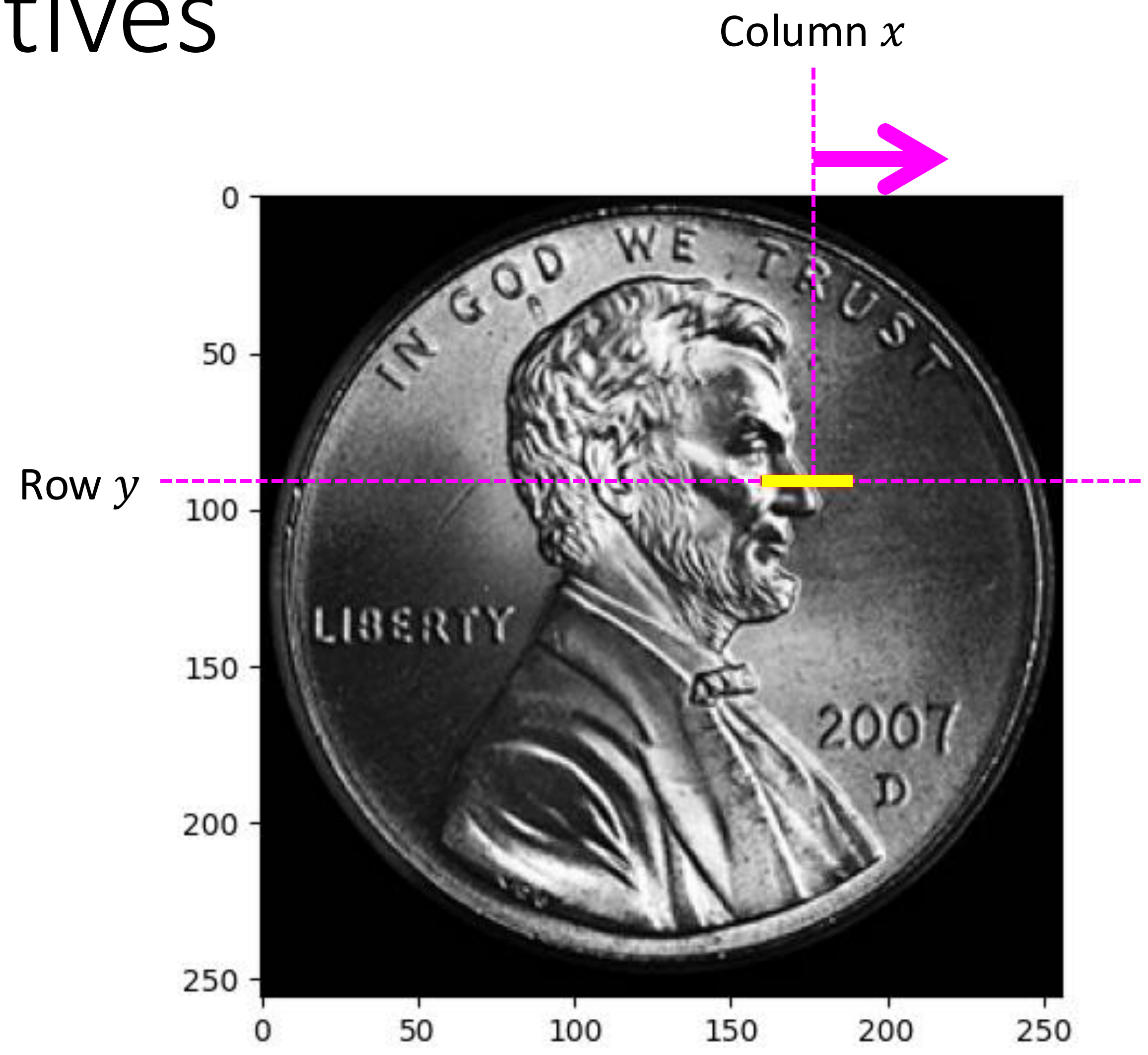
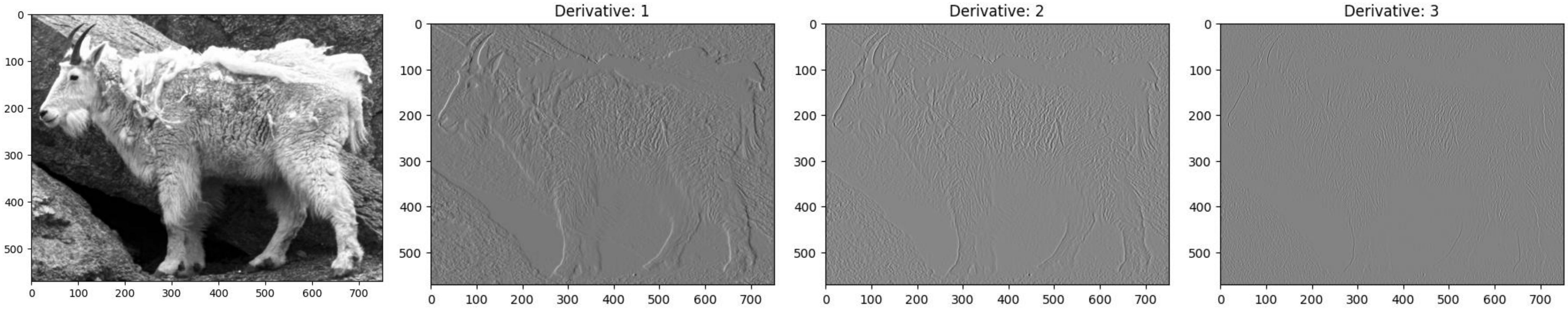


Image derivatives

Fitting a 3rd-order Taylor series using a 5-pixel patch



Summary

- 1D image patches
- Approximating 1D image patches via polynomials
- Computing image derivatives via fitting polynomials
- Least squares solution to a system of linear equations
- Weighted least squares