Polynomial Approximation

Computational Photography (CSCI 3240U)

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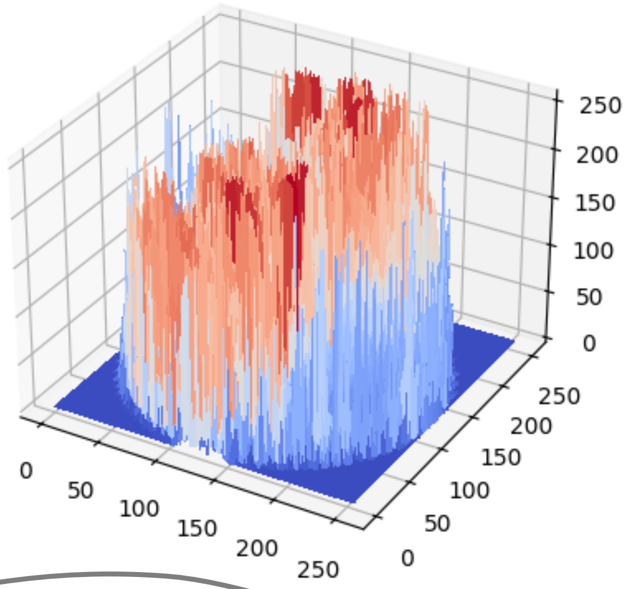
Today's lecture

- How to compute image derivatives by fitting polynomials to 1D image patches?
  - Taylor series expansion around a patch center
  - Least square fitting of a system of linear equations

Image as a surface in 3D

Consider a gray-scale image $I(x, y)$ then the height of the surface at (x, y) is $I(x, y)$. The surface passes through the 3D point $(x, y, I(x, y))$.



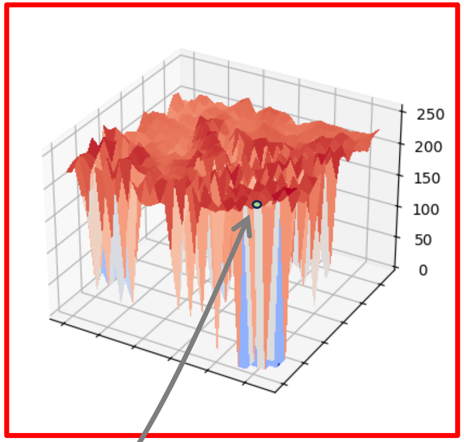
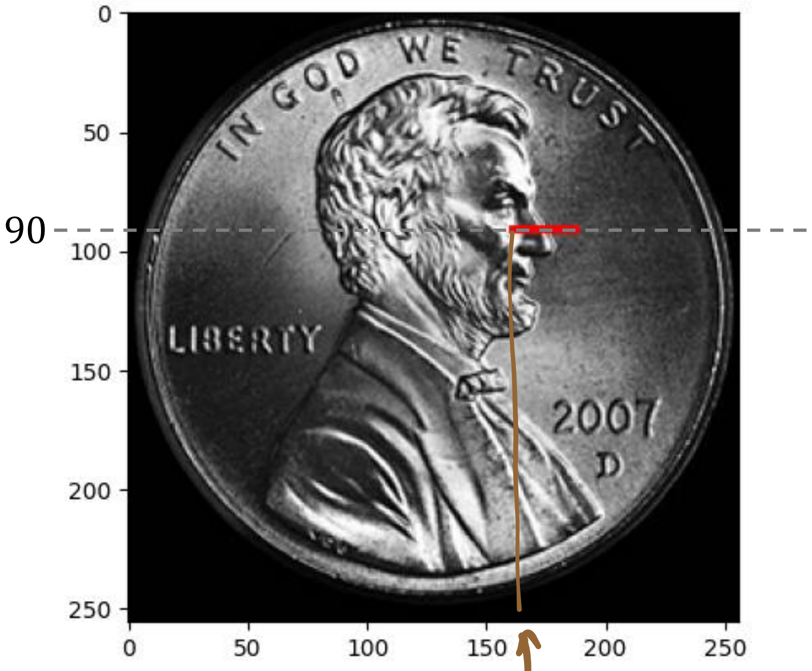
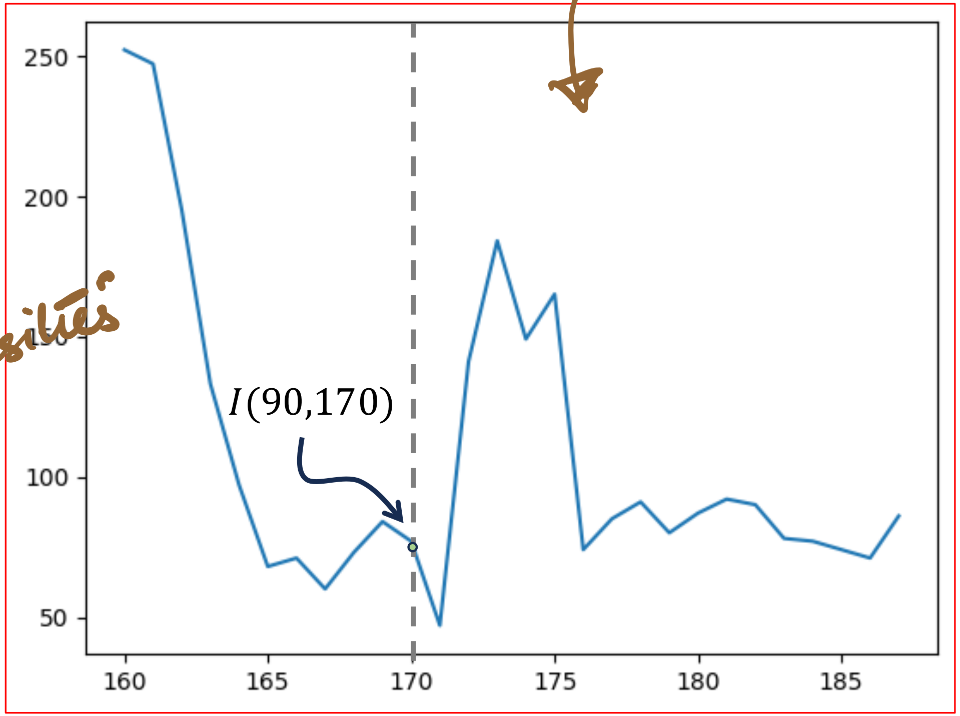


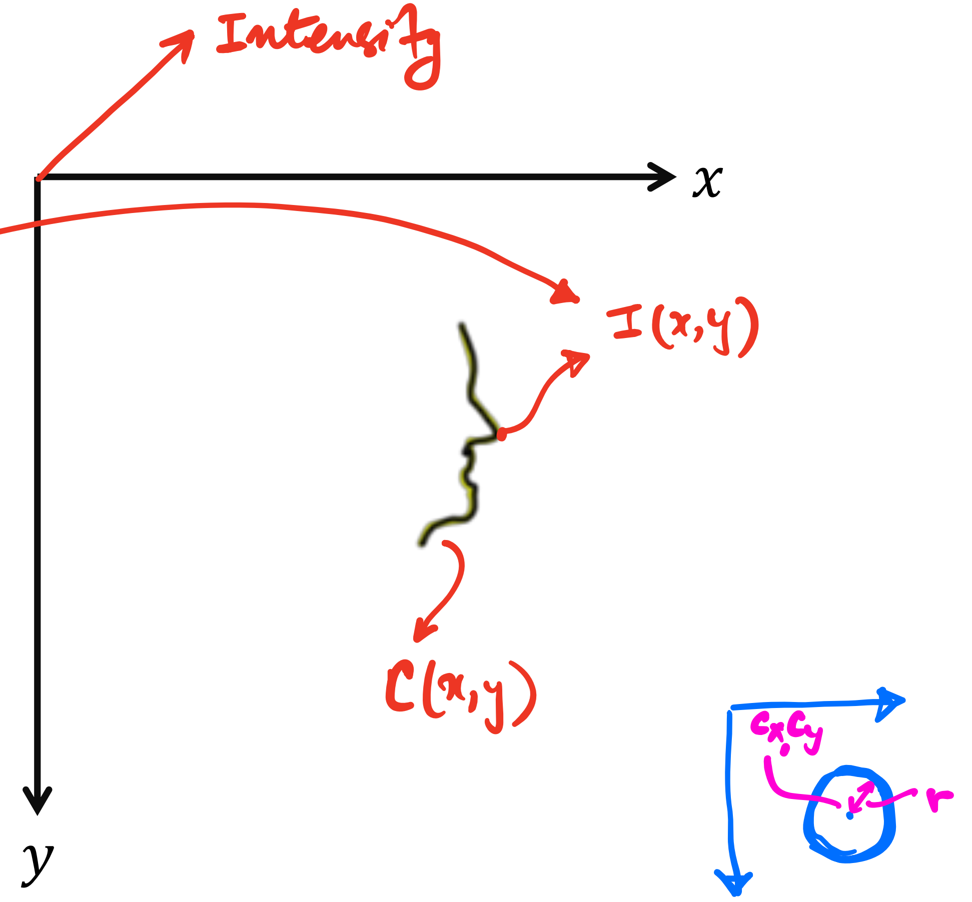
Image rows (or columns) as 2D graphs

row fixed at 90



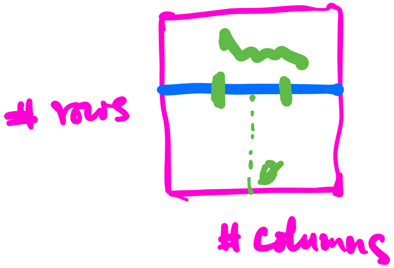


Paths as curves in 2D

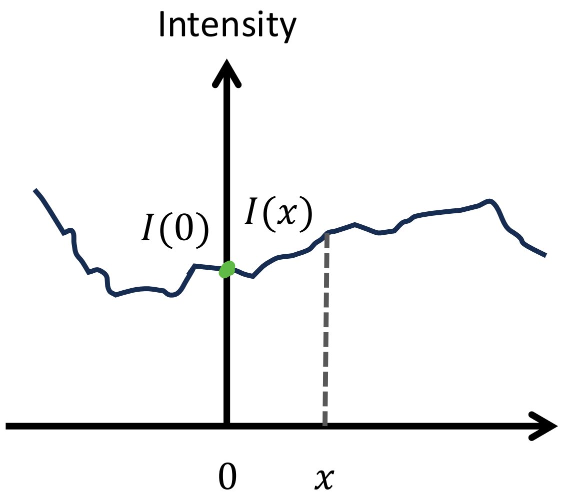


$$(x - c_x)^2 + (y - c_y)^2 = r^2$$

Image rows (or columns) as 2D graphs



Polynomial approximation



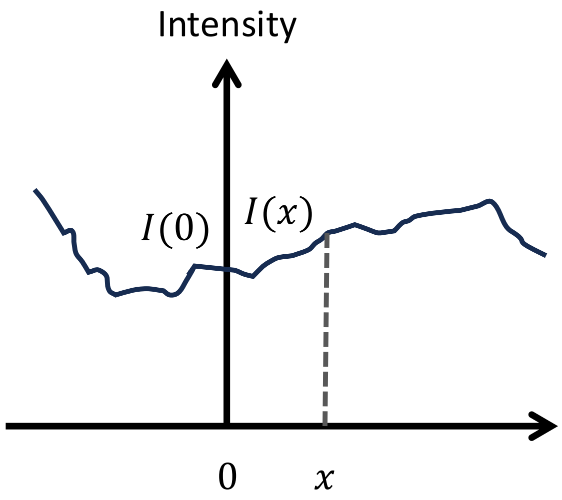
I(0) is provided.

Taylor series expansion of $I(x)$ near the "patch" center 0

$I(x) = ?$

Image rows (or columns) as 2D graphs

Polynomial approximation

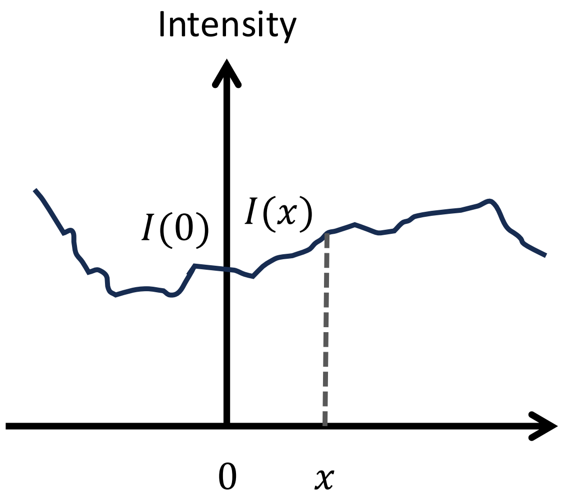


Taylor series expansion of $I(x)$ near the "patch" center 0

$I(x) = I(0)$

Image rows (or columns) as 2D graphs

Polynomial approximation

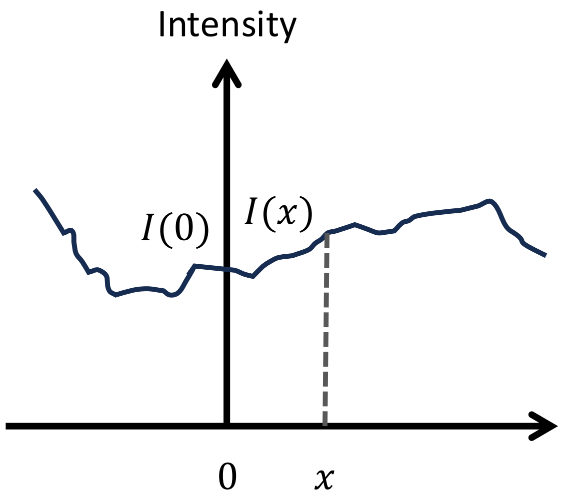


Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = I(0) + xI'(0)$$

Image rows (or columns) as 2D graphs

Polynomial approximation



Taylor series expansion of $I(x)$ near the “patch” center 0

1

$$I(x) = I(0) + xI'(0) + \frac{x^2}{2!} I''(0)$$

$0!$ $1!$

$f(x) = x^3 - 2x + 3$

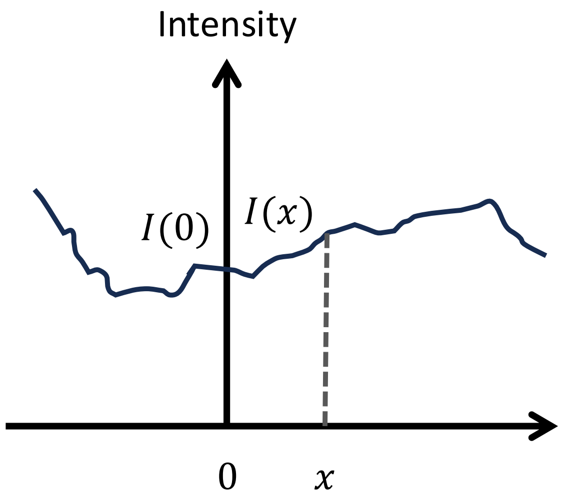
$\frac{df(x)}{dx} = f' = 3x^2 - 2$

$\frac{d^2f(x)}{dx^2} = f'' = 6x$

$7! = (7)(6)(5)(4)(3)(2)(1)$

Image rows (or columns) as 2D graphs

Polynomial approximation



Taylor series expansion of $I(x)$ near the "patch" center 0

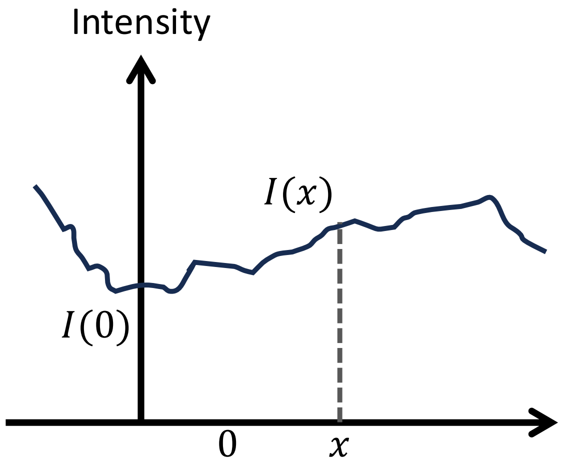
$$I(x) = I(0) + xI'(0) + \frac{x^2}{2!}I''(0) + \frac{x^3}{3!}I'''(0) + \cdots + \frac{x^n}{n!}I^{(n)} + R_{n+1}(x)$$

The residual $R_{n+1}(x)$ satisfies:

$$\lim_{x \to 0} R_{n+1}(x) = 0$$

Image rows (or columns) as 2D graphs

Polynomial approximation

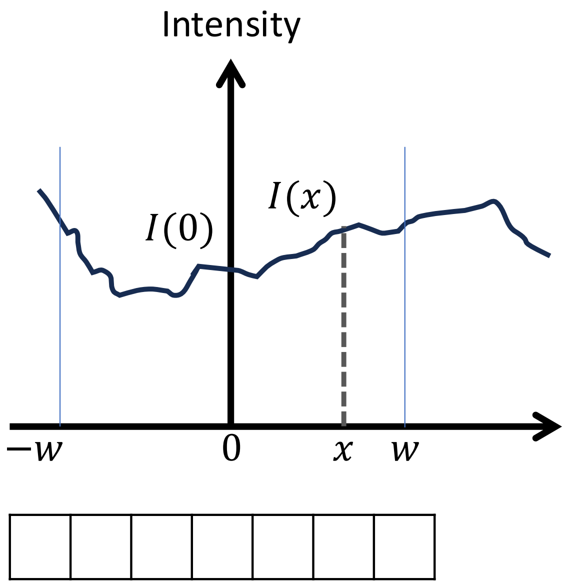


Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = \underbrace{I(0) + xI'(0) + \frac{x^2}{2!}I''(0) + \frac{x^3}{3!}I'''(0) + \cdots + \frac{x^n}{n!}I^{(n)}}_{\text{Nth order approximation}} + R_{n+1}(x)$$

For a given x, approximation depends on $(n+1)$ constants corresponding to the intensity derivative at the patch origin.

Polynomial approximation

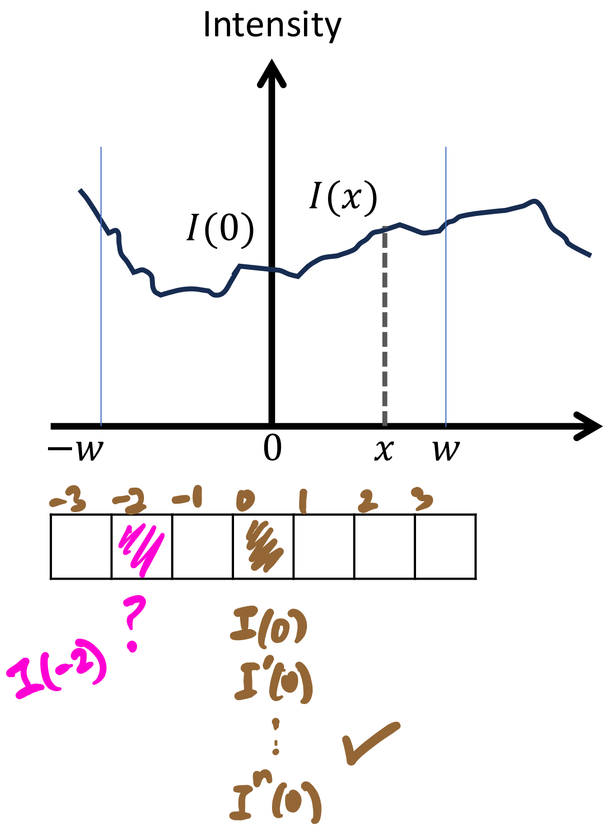


Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Re-write in matrix form

Polynomial approximation



Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Re-write in matrix form

dot-product

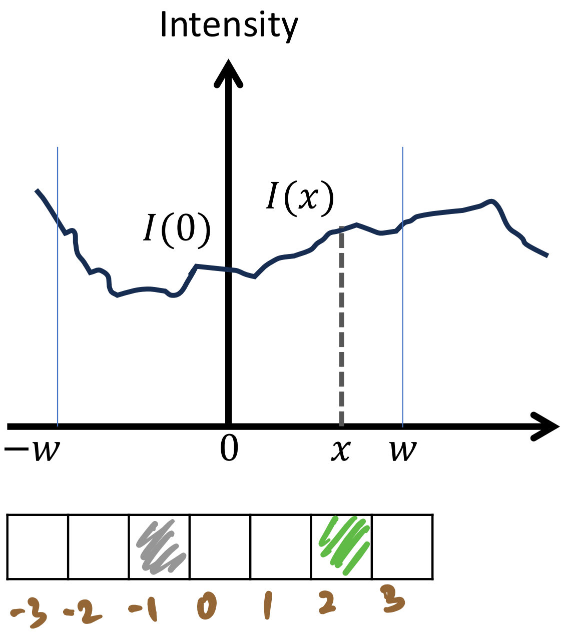
$$I(x) \approx \begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \ldots & \frac{1}{n!}x^n \end{bmatrix} \begin{bmatrix} I(0) \\ I'(0) \\ I''(0) \\ I'''(0) \\ \vdots \\ I^{(n)} \end{bmatrix}$$

$\in \mathbb{R}^{n+1}$

FIXED!

For notational simplicity, lets refer the vector of intensity and its derivatives as $\boldsymbol{d}$

Polynomial approximation



Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Example

Show the 0th order approximation

$I(x) \approx I(0)$

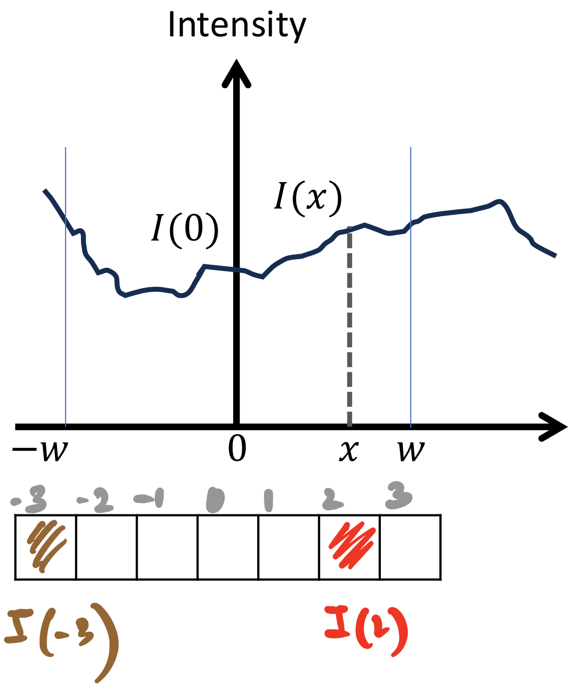
what is $I(2)$?

$I(2) = I(x)$

what is $I(-1)$?

$I(-1) = I(x)$

Polynomial approximation



Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Practice Question

Show the 1st and 2nd order approximations

$$I(x) = \underline{I(0)} + x\,\underline{I'(0)} + \frac{x^2}{2}\,\underline{I''(0)}$$

what is the value at I(2)?

$$I(2) = I(0) + 2I'(0) + 2I''(0)$$

what is the value at I(-3)?

$$I(-3) = I(0) - 3I'(0) + \frac{9}{2}I''(0)$$

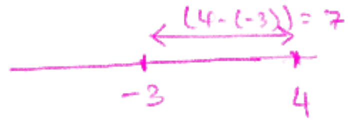
EXAMPLE PROBLEM :

Given $I(-3) = 4$, $I'(-3) = 2$, $I''(-3) = -1$ and $I'''(-3) = -1/2$

Estimate $I(4)$ using second-order Taylors series expansion.

Solution: $I(x) = I(0) + x\,I'(0) + \frac{x^2}{2} I''(0)$

$x = 4 - (-3) = 7$



$$I(4) = I(-3) + (4-(-3))\,I'(-3) + \frac{(4-(-3))^2}{(2)(1)} I''(-3)$$

$$= 4 + (7)(2) + \left(\frac{49}{2}\right)(-1)$$

$$= 4 + 14 - \frac{49}{2}$$

$$= 18 - 49/2$$

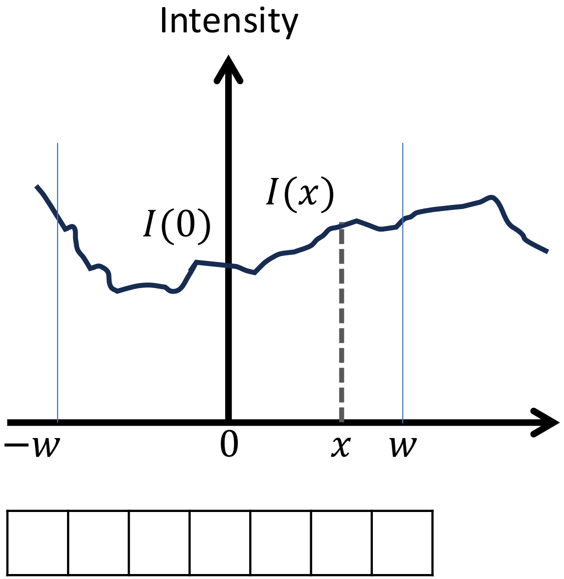
$$= \frac{36-49}{2}$$

$$= -13/2$$

$$= -6.5$$

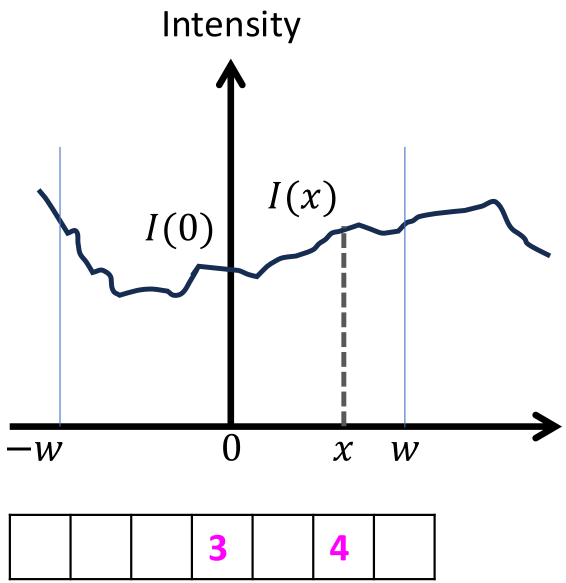
Compute derivatives at pixel 0 (i.e., the center of the pathc)

Fit a polynomial of degree n to the patch intensities



Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



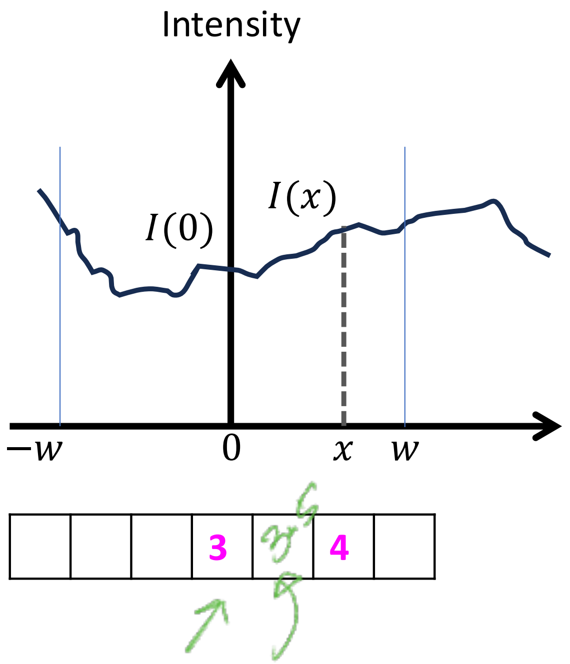
Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0)$$

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0)$$

Unknowns

highligted items are unknowns.

$$I(0) + (0)I'(0) + \frac{1}{2}(0)^2 I''(0) = 3$$

$$\Rightarrow I(0) = 3 \quad \text{—} \; (1)$$

$$I(0) + (2)I'(0) + \frac{1}{2}(2)^2 I''(0) = 4$$

$$\Rightarrow I(0) + 2I'(0) + 2I'(0) = 4 \quad \text{—} \; (2)$$

$$I(0) + (1)I'(0) + \frac{1}{2}I''(0) = 3.5$$

$$\Rightarrow I(0) + I'(0) + \frac{1}{2}I''(0) = 3.5 \qquad \text{③}$$

System of linear Equations:

$$I(0) = 3$$
$$I(0) + 2I'(0) + 2I''(0) = 4$$
$$I(0) + I'(0) + \frac{1}{2}I''(0) = 3.5$$

Re-write as $Ax = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 1 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} I(0) \\ I'(0) \\ I''(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3.5 \end{bmatrix}$$

Estimate derivatives at pixel location 0

Quiz:

(0, 5), (1, 4), (3, 2), and (10, 5)

Solution

$$ax^3 + bx^2 + cx + d = I(x)$$

Unknowns : a, b, c, and d.

Eq. for (0,5)

$$a(0)^3 + b(0)^2 + c(0) + d = 5$$

$\Rightarrow$ $d = 5$

Eq. for (1,4)

$$a(1)^3 + b(1)^2 + c(1) + d = 4$$

$\Rightarrow$ $a + b + c + d = 4$

Eq. for (3,2)

$$a(3)^3 + b(3)^2 + c(3) + d = 2$$

$\Rightarrow$ $27a + 9b + 3c + d = 2$

Eq. for (10,5)

$$a(10)^3 + b(10)^2 + c(10) + d = 5$$

$\Rightarrow$ $1000a + 100b + 10c + d = 5$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \\ 1000 & 100 & 10 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 5 \end{bmatrix}$$