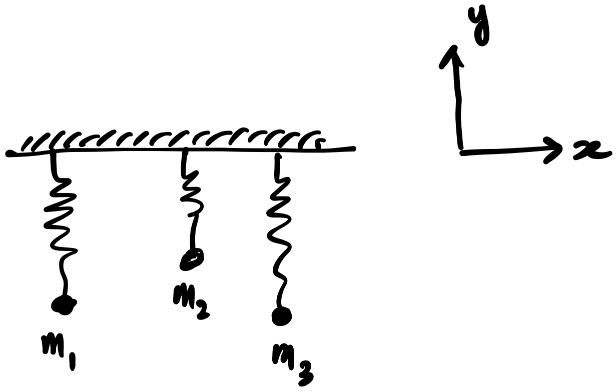
Jan 31, 2024

Continuous Systems

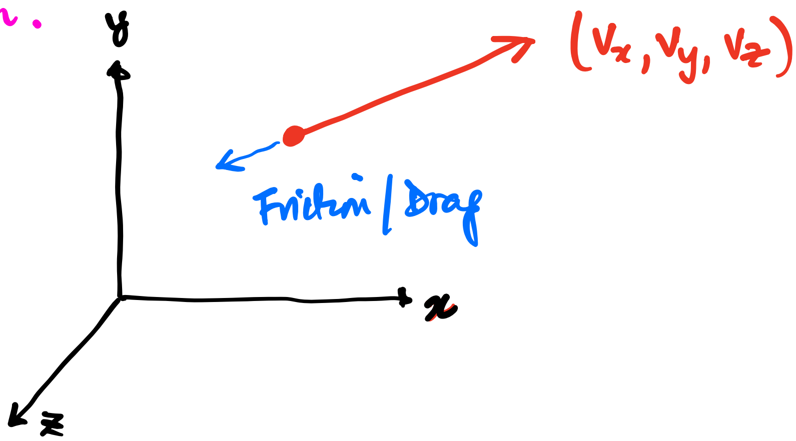
$F = ma$

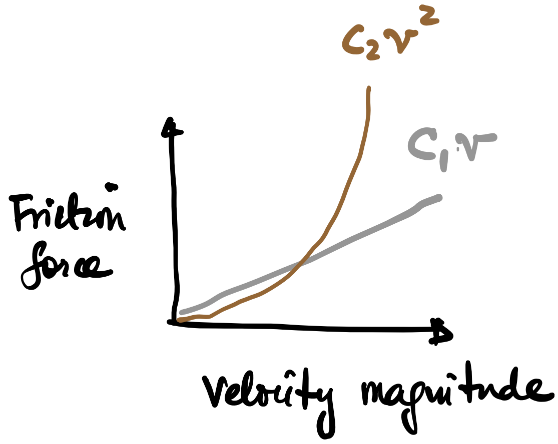
$\Rightarrow F = m \dfrac{d^2x}{dt^2}$

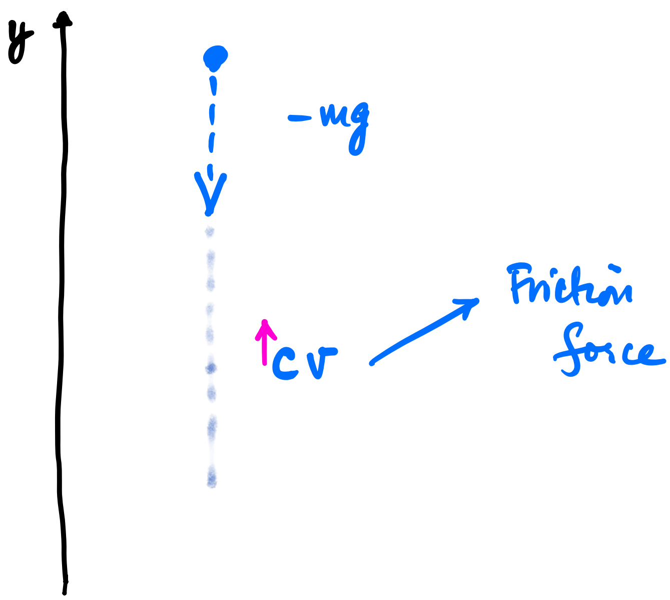
$F = -kx$



Friction.







Force due to friction will continue to increase till it equates mg. At which point the ball will have no force acting. The velocity will remain constant.

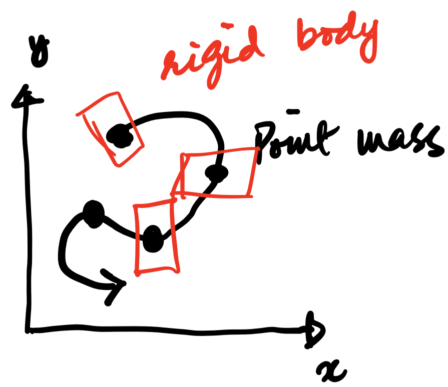
Terminal velocity

Thought Experiment

Modelling a coffee filter

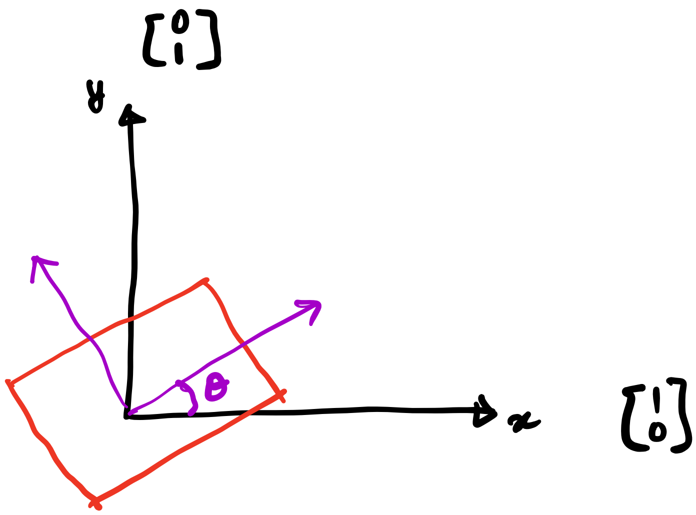
position (cm)	time (sec)
300	0
290	1
285	2
284	3

RIGID BODIES

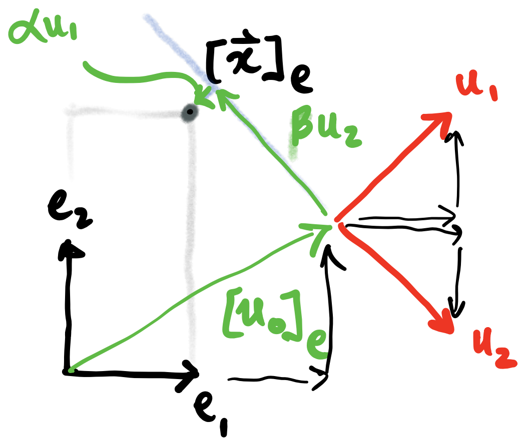


State of a rigid body in 2D

- $\vec{P} = (P_x, P_y)$
- $\vec{v} = (v_x, v_y)$
- θ orientation
- ω angular velocity



Coordinate Frames



$$[\vec{x}]_e = (1,2)$$
$$[\vec{x}]_u = (\alpha, \beta)$$

$$[\vec{x}]_e = e_1 + 2e_2$$

$$[\vec{x}]_e = [u_0]_e + \alpha[u_1]_e + \beta[u_2]_e$$

$$\Rightarrow \alpha[u_1]_e + \beta[u_2]_e = [\vec{x}]_e - [u_0]_e$$

$$\Rightarrow \begin{bmatrix}[u_1]_e & [u_2]_e\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = [\vec{x}]_e - [u_0]_e$$

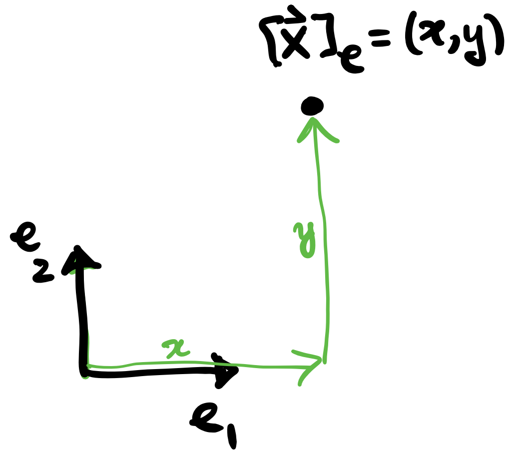
$$\Rightarrow \begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix}[u_1]_e & [u_2]_e\end{bmatrix}^{-1}\left([\vec{x}]_e - [u_0]_e\right)$$

point coordinates in the new coordinate system

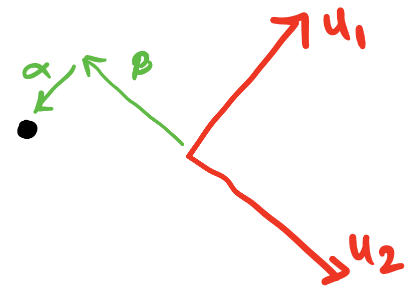
basis vectors of the new coordinate systems

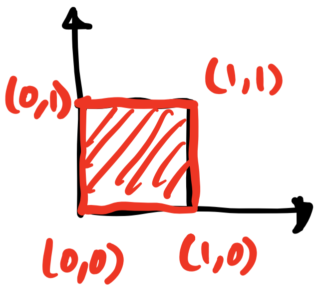
point in the old coordinate system

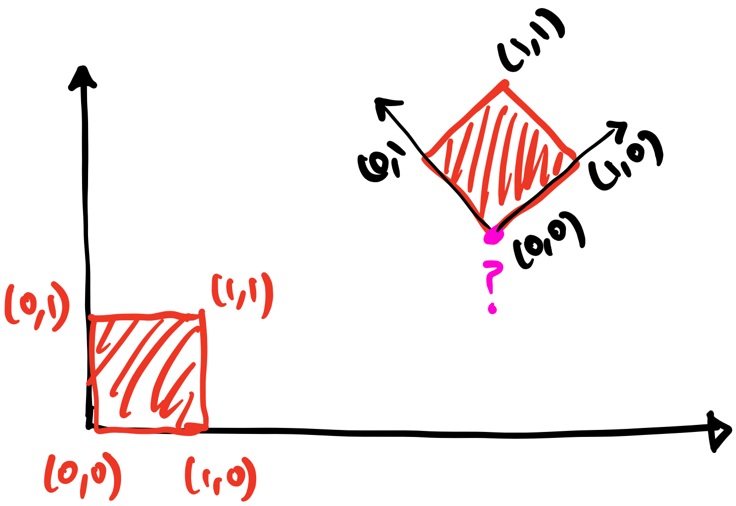
origin of the new coordinate system



$$[\vec{x}]_e = x\vec{e}_1 + y\vec{e}_2$$







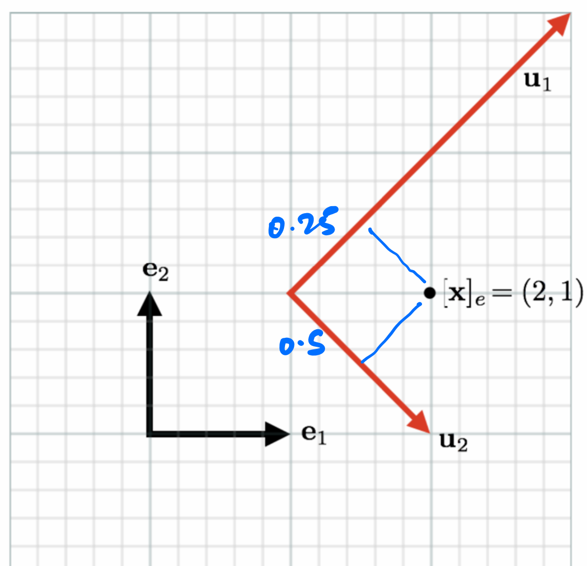
Example:

$[u_0]_e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$[u_1]_e = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$[u_2]_e = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$[\vec{x}]_e = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



$[\vec{x}]_e$ $[u_0]_e$

We know:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 - 1 \\ 1 - 1 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.25 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$[u_1]_e$ $[u_2]_e$

$$= \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}$$

Rigid Bodies.

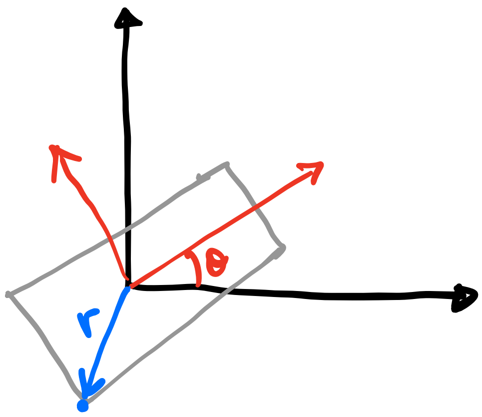
position : $\vec{p}$

velocity : $\vec{v}$

angular velocity : ω

orientation : θ

$$\frac{d\theta}{dt} = \omega$$



$\vec{v} = \vec{r}\omega$

Forces acting on the body : $F = ma$

$$\frac{dP}{dt} = F$$

force

rate of change of the linear Momentum

$P = mv$

$$\frac{dmv}{dt} = F$$

$$\Rightarrow m\frac{dv}{dt} = F$$

$$\Rightarrow ma = F$$

Torque acting on the rigid body

$$\frac{dL}{dt} = N$$

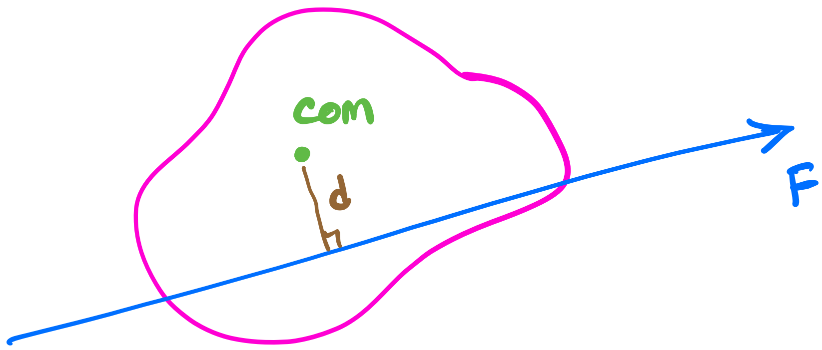
torque

rate of change of angular momentum

inertia tensor

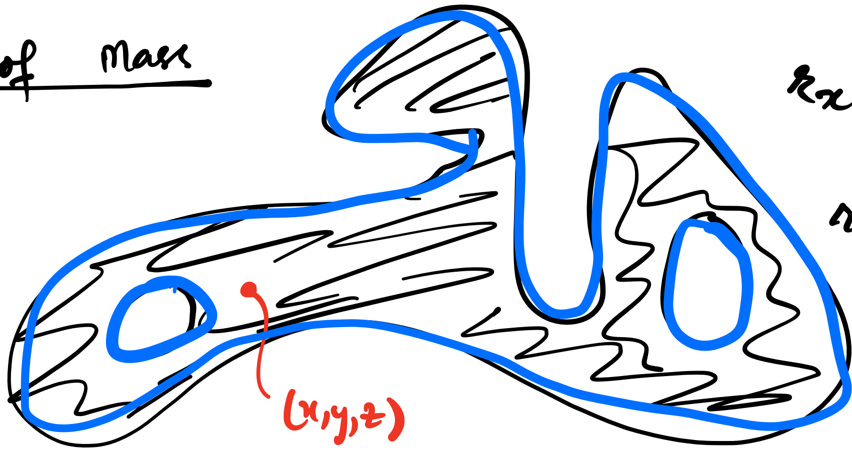
angular velocity

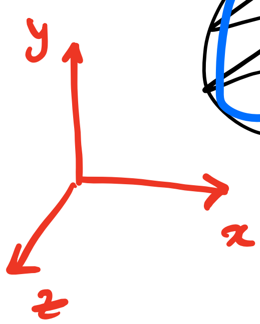
$L = I\omega$



Torque $= \vec{d} \times \vec{F}$

Center of Mass





$$r_x = \frac{1}{M}\int \rho(x,y,z)\, x\, dV$$

$$r_y = \frac{1}{M}\int \rho(x,y,z)\, y\, dV$$

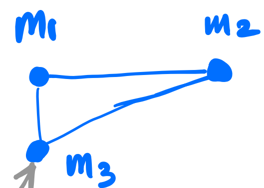
$$r_z = \frac{1}{M}\int \rho(x,y,z)\, z\, dV$$

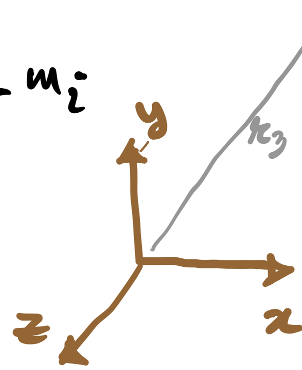
Discretize com calculation:

$$r_x = \sum_i m_i x_i \Big/ \sum_i m_i$$

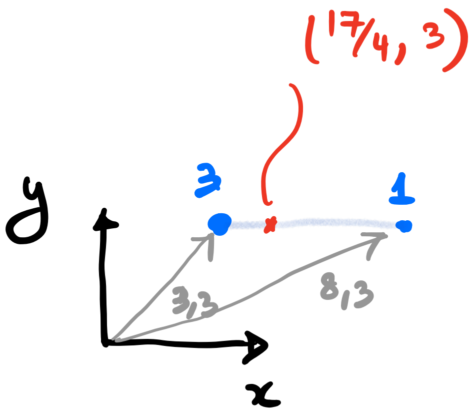
$$r_y = \sum_i m_i y_i \Big/ \sum_i m_i$$

$$r_z = \sum_i m_i z_i \Big/ \sum_i m_i$$





$$\sum_i m_i = m_1 + m_2 + m_3$$



$$R_x = \frac{(3)(3) + (8)(1)}{3+1} = \frac{9+8}{4} = \frac{17}{4}$$

$$R_y = \frac{(3)(3) + (3)(1)}{3+1} = \frac{9+4}{4} = \frac{12}{4} = 3$$

Inertia Tensor

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$$

$$I_{yy} = \sum_i m_i (x_i^2 + z_i^2)$$

$$I_{zz} = \sum_i m_i (x_i^2 + y_i^2)$$

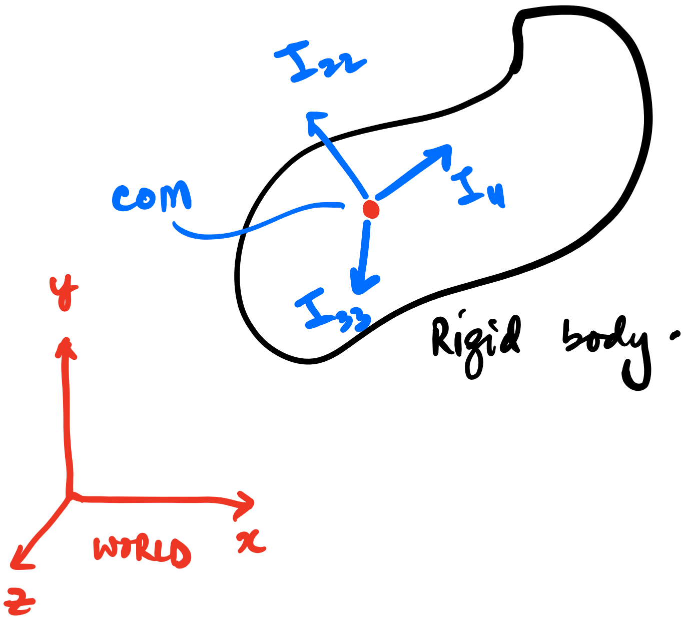
$$I_{xy} = I_{yx} = \sum_i m_i x_i y_i$$

$$I_{yz} = I_{zy} = \sum_i m_i z_i y_i$$

$$I_{xz} = I_{zx} = \sum_i m_i z_i x_i$$

Inertia Tensor in Body Coordinate System

$$I_{body} = \begin{bmatrix} I_{11} & & \\ & I_{22} & \\ & & I_{33} \end{bmatrix} \longrightarrow I_{body}^{-1} = \begin{bmatrix} 1/I_{11} & & \\ & 1/I_{22} & \\ & & 1/I_{33} \end{bmatrix}$$



KEY IDEAS

1. Angular effects.
2. MASS IS DISTRIBUTED
   - Inertia tensor
3. Change of basis
4. F=ma $\longrightarrow$ $\frac{dP}{dt} = F$

$$\frac{dL}{dt} = N$$

$$N = \vec{d} \times \vec{F}$$

$$P = mv$$

$$L = I\omega$$

$$\omega = \frac{d\theta}{dt}$$

center of mass.

Exercise.