Continuous Systems Simulations

RECAP

Ordinary differential equations

$$\frac{d^2x}{dt^2} = 2$$

t (usually time) is the independent variable

Solving ODEs (Integration).

1. Integrate once

$$\frac{dx}{dt} = 2t + c_1$$

2. Integrate again

$$x = t^2 + c_1 t + c_2$$

We can use this equation to find the value of x at specific values of t. However, we do not know the values of c_1 and c_2.

Constants of integration.

Use initial conditions

$$x(0) = 7$$

$$\frac{dx}{dt}(0) = 3$$

OR

boundary conditions

$$x(0) = 7$$

$$x(72) = -3$$

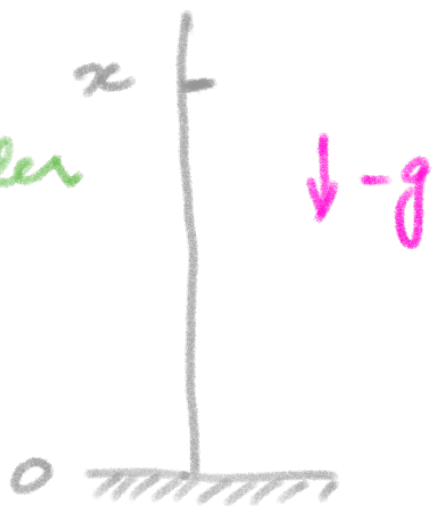
To solve for c_1 and c_2.

Reducibility

Replace a single N^{th}-order ODE with N 1st-order ODEs.

Ex.

$$\underbrace{\frac{d^2x}{dt^2} = -g}_{\text{2nd order}}$$



can be re-written as

$$\underbrace{\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -g \end{aligned}}_{\text{Two 1st order ODEs}}$$

We can write the update rules as follows

$$\Delta x = v \Delta t$$

$$\Delta v = -g \Delta t$$

using the two 1st-order equations developed above.

But we are interested in finding the values of x and v for a specific time t.

This is what we really need; however, we cannot compute it without also knowing v.

$$x(t+\Delta t) = x(t) + v(t)\,\Delta t$$
$$v(t+\Delta t) = v(t) - g\,\Delta t$$

current values for x and v

Updated values for x and v (after time Δt)

Initialization : $t=0$, $x(0)$, $v(0)$

do

  compute $x(t+\Delta t)$, $y(t+\Delta t)$

  $t = t + \Delta t$

while (1);

choice of Δt.

NEWTON'S EQ. OF MOTION.

$$F = ma$$

$$\Rightarrow F = m\frac{d^2x}{dt^2}$$ 2nd Order

Force

mass

position

REDUCIBILITY

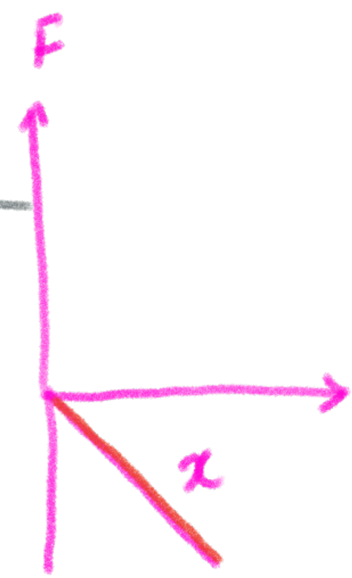
$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \frac{F}{m}$$

MASS-SPRING SYSTEM

rest length = 0.

Hook's Law: $F = -kx$

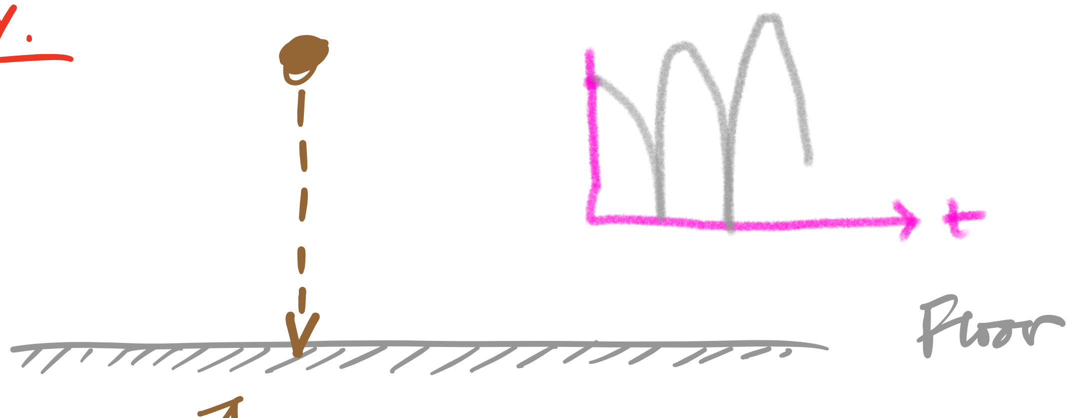


Use this to capture the position of mass m over time.

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \frac{-kx}{m}$$

BALL FALLING UNDER THE INFLUENCE OF GRAVITY.



penetrates the floor.

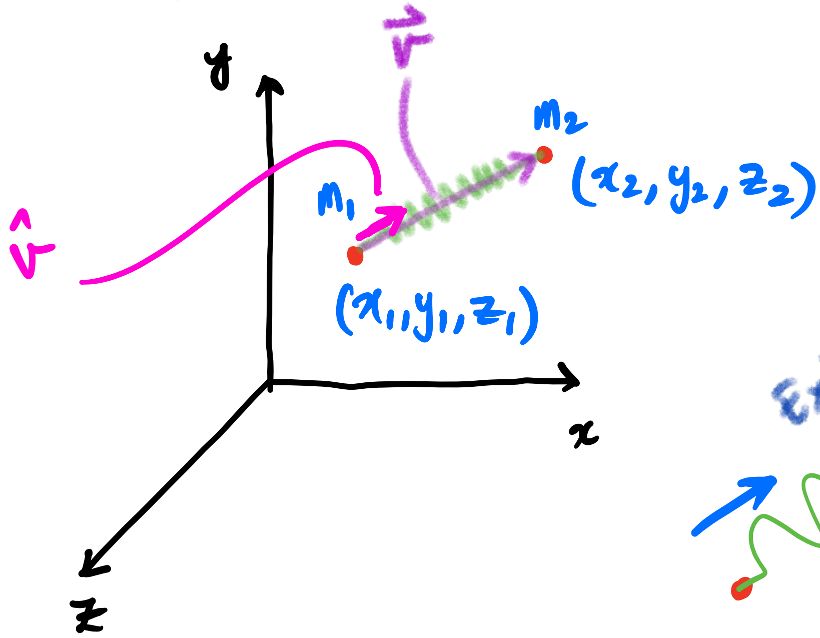
Euler integration is inaccurate. It requires us to use very small Δt.

Use RK4

↳ state variables: x, v

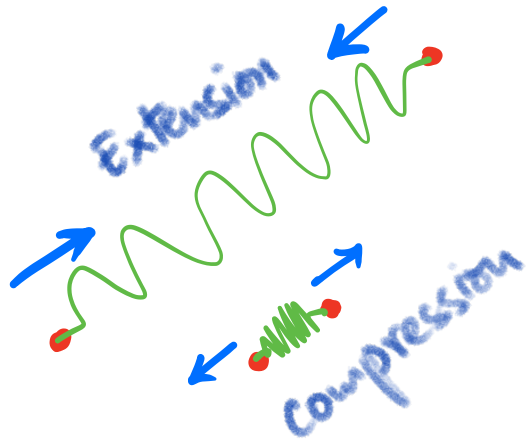
variables tracked by ODEs.

SPRINGS IN 3D



Rest length $= l_r$

Spring constant $= k$



Current length:

$$l_c = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Deformation:

$$d = l_c - l_r$$

NORM

Apply Hook's Law:

$$F = -kd$$

Vector (spring axis):

$$\vec{v} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$

Unit vector along the spring axis:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}}{l_c}$$

Force on mass m_1:

$$F_1 = +kd\,\hat{r} \in \mathbb{R}^3$$

$d > 0$ if extended force on m_1 is along $\hat{r}$.

$d < 0$ if compressed.

Force on mass m_2:

$$F_2 = -kd\,\hat{r} \in \mathbb{R}^3$$

ASIDE: Eq. of Motion in 1D

$$F = ma$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{F}{m}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = v \\[2ex] \dfrac{dv}{dt} = \dfrac{F}{m} \end{cases}$$

Velocity of mass m_1: (v_1^x, v_1^y, v_1^z)

Apply in 3D

For mass m_1:

$$\frac{dv_1^x}{dt} = \frac{+kd\,\hat{r}^x}{m_1} \qquad \frac{dx_1}{dt} = v_1^x$$

$$\frac{dv_1^y}{dt} = \frac{+kd\,\hat{r}^y}{m_1} \qquad \frac{dy_1}{dt} = v_1^y$$

$$\frac{dv_1^z}{dt} = \frac{+kd\,\hat{r}^z}{m_1} \qquad \frac{dz_1}{dt} = v_1^z$$

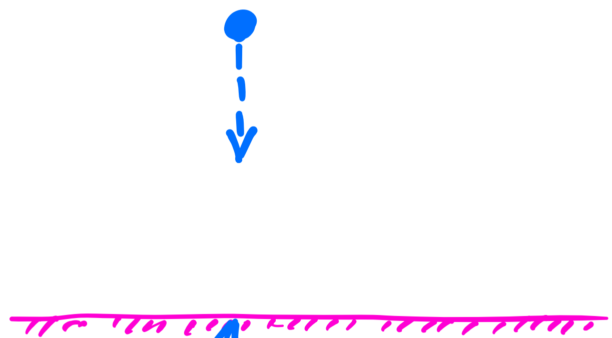
We will write the 6 equations for m_2.

State variables:

$x_1, y_1, z_1, v_1^x, v_1^y, v_1^z$

$x_2, y_2, z_2, v_2^x, v_2^y, v_2^z$

Ball that is falling under gravity.



FLOOR

1. collision detection

   check height x of the ball.

   $\boxed{x < 0}$

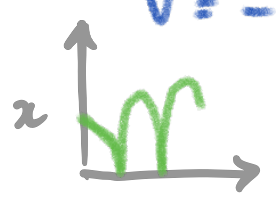
   collision

2. Collision response

   Set $x = 0$

   $v = -v$

   imparts energy to the system.



Assume no friction

Initially all the energy of the ball is Potential Energy.

$$P.E. = mgh$$

initial height at which the ball was released.

At height ~~0~~ all the P.E. has converted to K.E.

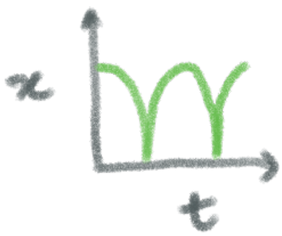
$$K.E. = \frac{1}{2} m v^2$$

velocity of the ball.

At Floor: $mgh = \frac{1}{2} m v^2$

$$\Rightarrow v^2 = 2gh$$

$$\Rightarrow v = \sqrt{2gh}$$



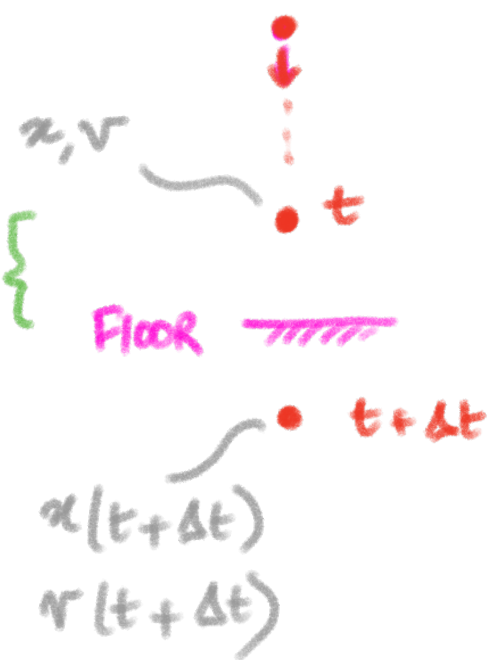
Collision response 2: set $x = 0$

$v = \sqrt{2gh}$

flip v dir.

Assume that there is friction.

We cannot use method 2 above.



How much time it takes for the ball to cover x distance when moving with velocity v.

distance

$$\boxed{s = vt}$$

time

velocity

$$t_c = \frac{x}{v}$$ → Approx. Assumes that velocity is constant.

Collision response 3:

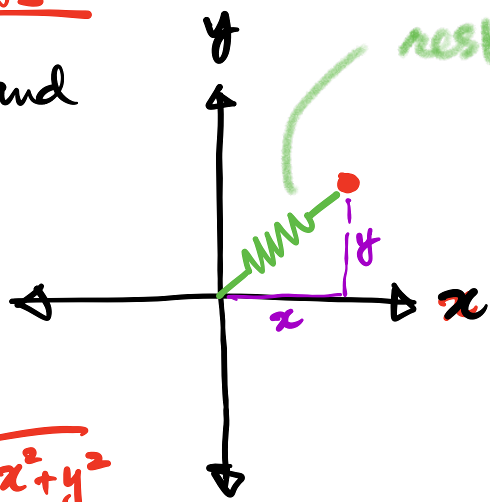
set $x = x(t + t_c)$

$v = -v(t + t_c)$

SIMPLIFICATIONS

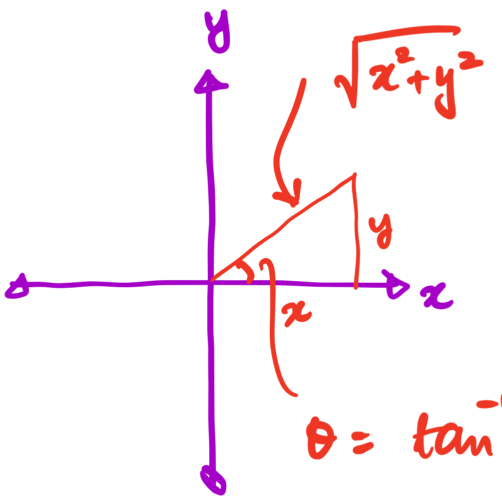
2D Elastic Band

rest length = 0



$$\underbrace{\begin{aligned} F_x &= -kx \\ F_y &= -ky \end{aligned}}$$

Direct application of Hook's law in 1D



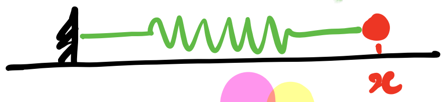
$$\theta = \tan^{-1}(y/x)$$

$$F_x = -k\left(\sqrt{x^2+y^2}\right)\cos\theta$$

$$F_y = -k\left(\sqrt{x^2+y^2}\right)\sin\theta$$

MASS-SPRING-DAMPER

rest length = 0



$F = -kx - cv$

velocity

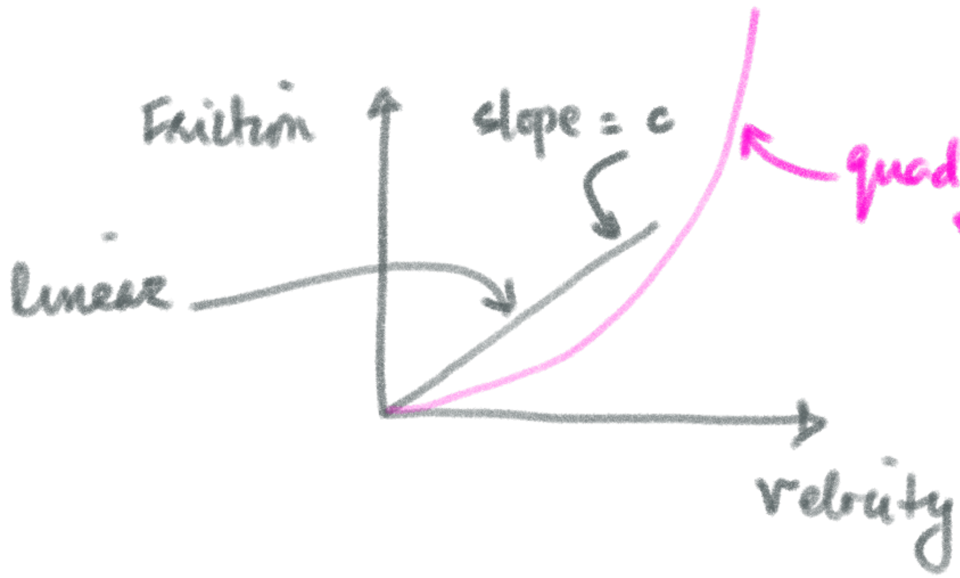
co-efficient of friction.

Linear Model of Friction.

Force $\longrightarrow$ $F = ma$

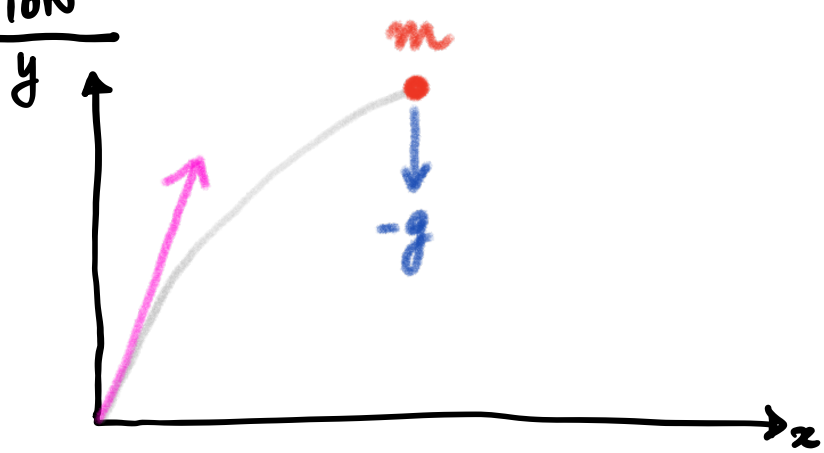
$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -\frac{k}{m}x - \frac{c}{m}v$$



quadratic model of friction.

PROJECTILE MOTION



State: x, y, v_x, v_y

if no friction

if friction

Forces in x-direction: 0 OR $-\gamma v_x$

coeff. of friction.

Forces in y-direction: $-g - \gamma v_y$

$$\frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = v_y$$

$$\frac{dv_x}{dt} = -\frac{\gamma v_x}{m}$$

$$\frac{dv_y}{dt} = -\frac{(g + \gamma v_y)}{m}$$

$$F = ma$$

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \frac{F}{m}$$

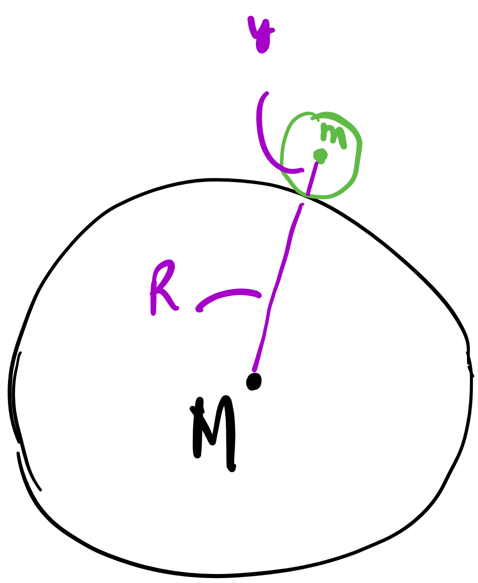
$g = 9.8$ m/s^2

Newton's law of Universal Gravitation

$$\boxed{F = \frac{GMm}{(R+y)^2}}$$

$G = 6.674 \times 10^{-11}$ N(m^2/kg^2)

$$g \approx \frac{GM}{R^2}$$



$$\frac{dx}{dt} = v_x$$

$$\frac{dy}{dt} = v_y.$$

$$\frac{dv_x}{dt} = -\frac{\gamma v_x}{m}$$

$$\frac{dv_y}{dt} = -\frac{\gamma v_y}{m} - \frac{GMm}{(R+y)^2}$$

We cannot solve this analytically.