Rigid Body Dynamics (Unconstrained)

Particles:

State vector of a particle $Y_t = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$

State vector of N particles

$$Y_t = \begin{pmatrix} x_1(t) \\ v_1(t) \\ \vdots \\ x_n(t) \\ v_n(t) \end{pmatrix}$$

To simulate the motion of these particles, we need to know the force acting on them.

Change in Y_t over time is

$$\frac{d}{dt} Y(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ F(t)/m \end{pmatrix}$$ ← For a single particle.

Easily extended for n particles. If we know how to simulate a single particle, we can simulate n particles.

Rigid Body.

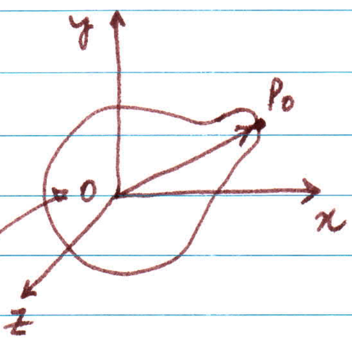
Has both position and orientation.

represent position of a rigid body using a translation $x(t)$.

How to represent the orientation of a rigid body?

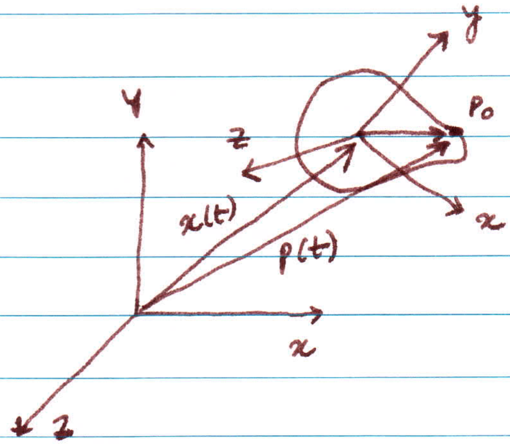
$R(t) \in \mathbb{R}^{3\times 3}$, a rotation matrix.

$x(t)$ and $R(t)$ are referred to as the spatial variables of the rigid body.



Body space.

lets refer to this as the centre of mass (COM)



World Space.

$p(t)$ is point P_0 expressed in the world space.

$$p(t) = x(t) + R(t)P_0$$

$R(t) = ?$

x-axis of the body coordinate system expressed in the body coordinate system is $(1,0,0)$

y-axis $\longrightarrow$ $(0,1,0)$

z-axis $\longrightarrow$ $(0,0,1)$.

Let $R(t) = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$.

$\Rightarrow \quad R(t) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Observation: the first column of the rotation matrix $R(t)$ is the x-axis of the body coordinate system expressed in ~~the~~ the world coordinate system.

$\therefore R(t) = \begin{bmatrix} r_{xx} & r_{yx} & r_{zx} \\ r_{xy} & r_{yy} & r_{zy} \\ r_{xz} & r_{yz} & r_{zz} \end{bmatrix}$.

Linear Velocity:

$v(t) = \dot{x}(t)$.

Imagine that the ~~body~~ orientation of the body is fixed.

Angular Velocity:

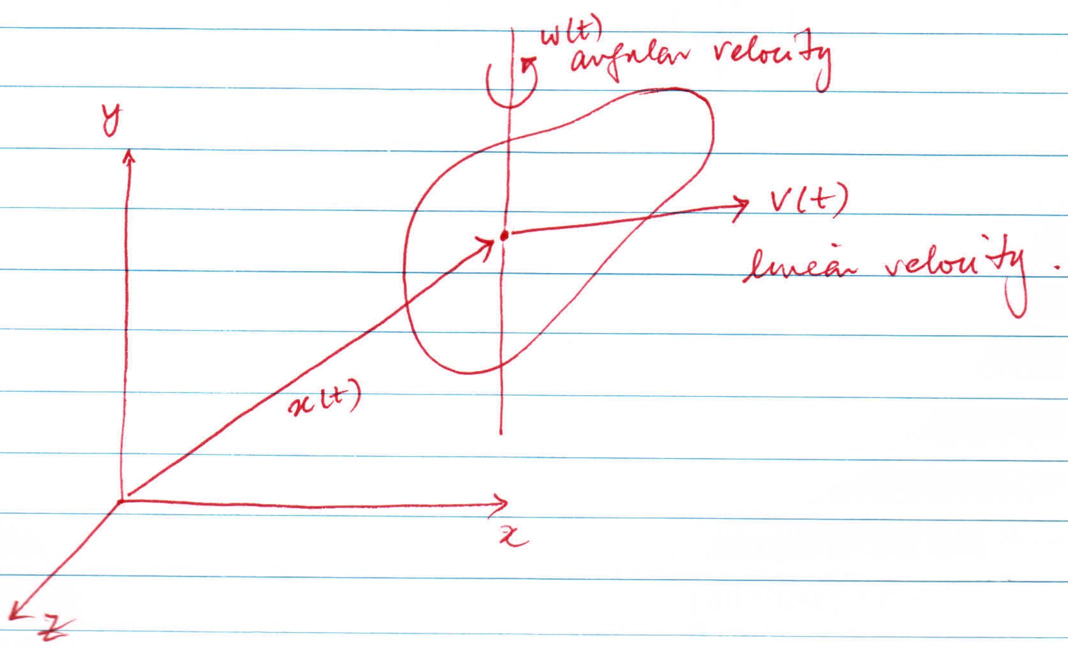
Imagine that we fixed the COM. Further assume that the body is spinning about an axis that passes through the COM.

otherwise COM will itself be moving.

We describe this spin as a vector $w(t)$.

direction gives the direction of axis about which the body is rotating.

magnitude tells how fast the body is rotating. (revolutions / time)

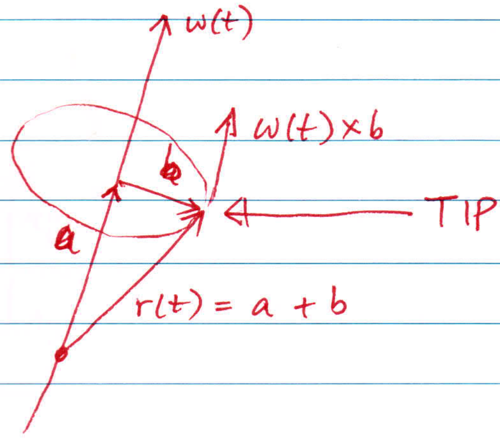


Relationship between $R(t)$ and $\omega(t)$.

lets consider a vector $r(t)$ that is rigidly attached to the rigid body.

↓

$r(t)$ is not affected by the translational effects.



what is the instantaneous velocity of TIP?

$|\omega(t) \times b| = |\omega(t)||b|$

∴ We can write $\dot{r}(t) = \omega(t) \times r(t)$

↓

Lets apply this to $R(t)$.

$$\dot{R} = \left[\omega(t) \times \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \quad \omega(t) \times \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \quad \omega(t) \times \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \right]$$

Given a vector $a = (a_x, a_y, a_z)$.

Define

$$a^* = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}.$$

then $a \times b = a^* b$.

So,

$$\dot{R} = \left(\omega(t)^* \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \quad \omega(t)^* \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \quad \omega(t)^* \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \right).$$

$$\Rightarrow \quad \dot{R}(t) = \underbrace{\omega(t)^* R(t)}_{\text{matrix-matrix multiplication}}$$

Mass of a body.

Consider ~~N~~ small particles : $m_1 \ldots m_N$, each at position r_{0i}, $1 \leq i \leq N$.

Total mass of the body : $M = \sum_{i=1}^{N} m_i$

Position of the i^{th} particle ?

$$r_i(t) = R(t)\, r_{0i} + x(t) .$$

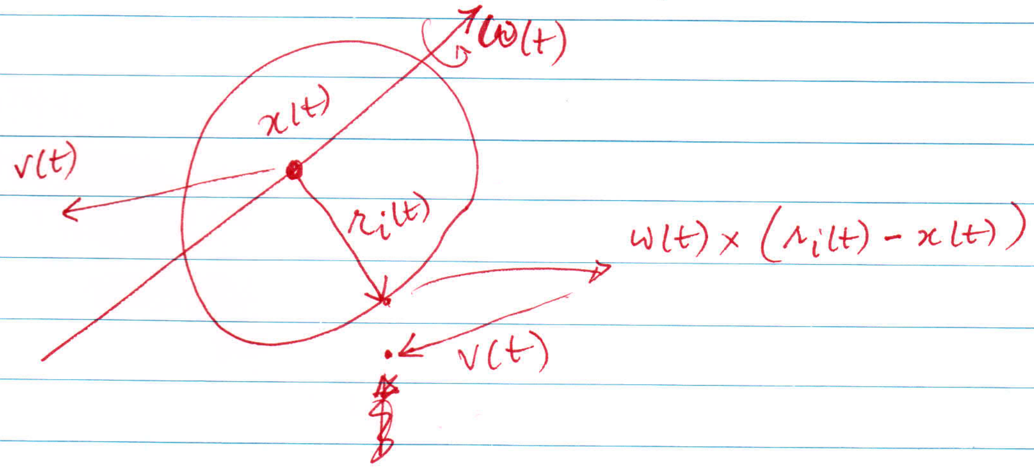
Velocity of the i^{th} particle ?

$$\dot{r}_i(t) = \omega(t)^{*} R(t)\, r_{0i} + v(t)$$

$$= \omega(t)^{*}\big(R(t)\, r_{0i} + x(t) - x(t)\big) + v(t)$$

$$= \omega(t)^{*}\big(r_i(t) - x(t)\big) + v(t)$$

$$= \underbrace{\omega(t) \times \big(r_i(t) - x(t)\big)}_{\substack{\text{rotational} \\ \text{(angular)} \\ \text{component.}}} + \underbrace{v(t)}_{\substack{\text{linear} \\ \text{component}}}$$



Dynamics of a rigid body.

Goal: separate into linear and angular components.

center of mass in the world space

$$= \frac{\sum m_i r_i(t)}{M}$$

We often use the center of mass coordinate system.
It means that in the body space.

$$\frac{\sum m_i r_{0i}}{M} = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \longrightarrow \sum m_i r_{0i} = 0$$

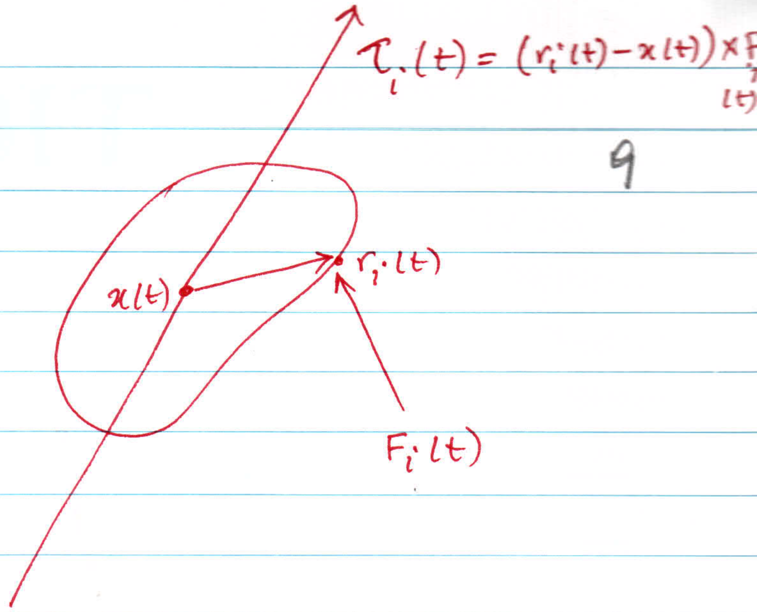
We have been implicitly assuming $x(t)$ to be the COM of the body. Is it true?

$$\frac{\sum m_i r_i(t)}{M} = \frac{\sum m_i \left(R(t) r_{0i} + x(t)\right)}{M}$$

$$= \frac{R(t) \sum m_i r_{0i} + \sum m_i x(t)}{M}$$

$$= x(t) \frac{\sum m_i}{M}$$

$$= x(t).$$



Force and Torque.

$$\underbrace{\tau_i(t)} = (r_i(t) - x(t)) \times F_i(t)$$

Differs from ~~f~~ force. The torque acting on a particle also depends upon the distance of the particle from the center of mass.

Total external force : $F(t) = \sum F_i(t)$

Total torque : $\tau(t) = \sum \tau_i(t) = \sum_i (r_i(t) - x(t)) \times F_i(t).$

Linear Momentum.

$$p = mv$$

total linear momentum

$$P(t) = \sum m_i \dot{r}_i(t)$$

$$\dot{r}_i(t) = v(t) + \omega(t) \times (r_i(t) - x(t))$$

$$P(t) = \sum m_i \left(v(t) + \omega(t) \times (r_i(t) - x(t))\right)$$

$$= \sum m_i v(t) + \omega(t) \times \underbrace{\sum m_i (r_i(t) - x(t))}$$

$$\sum m_i (R(t) r_{0i} + \cancel{x(t)} - \cancel{x(t)})$$

$$= R(t) \sum m_i r_{0i}$$

$$= 0$$

$$P(t) = \sum m_i v(t).$$

$$\Rightarrow v(t) = \frac{P(t)}{M}$$

$$\Rightarrow \dot{v}(t) = \frac{\dot{P}(t)}{M} = \frac{F(t)}{M}$$

Newton's Second Law.

Angular Momentum:

Defined by the equation

$$L(t) = I(t)\,\omega(t)$$

angular momentum

inertia tensor
a 3x3 matrix.

Also $\dot{L}(t) = \tau(t)$ ⇐ describes the relationship between torque and inertia tensor.

$$I(t) = \sum \begin{pmatrix} m_i(r'^2_{iy} + r'^2_{iz}) & -m_i r'_{ix} r'_{iy} & -m_i r'_{ix} r'_{iz} \\ -m_i r'_{iy} r'_{ix} & m_i(r'^2_{ix} + r'^2_{iz}) & -m_i r'_{iy} r'_{iz} \\ -m_i r'_{iz} r'_{ix} & -m_i r'_{iz} r'_{iy} & m_i(r'^2_{ix} + r'^2_{iy}) \end{pmatrix}$$

$$I(t) = R(t)\, I_{body}\, R(t)^T$$

defined in the body space so is constant.

Also, $I^{-1}(t) = R(t)\, I^{-1}_{body}\, R(t)^T$

$R(t)^T = R(t)^{-1}$

$(R(t)^T)^T = R(t)$.

Also a constant during simulation.

Rigid Body Equations of Motion:

$$Y(t) = \begin{pmatrix} x(t) \\ R(t) \\ P(t) \\ L(t) \end{pmatrix}$$

$$v(t) = \frac{P(t)}{M}, \quad I(t) = R(t)\, I_{body}\, R(t)^T, \quad w(t) = I(t)^{-1} L(t)$$

$$\frac{dY(t)}{dt} = \begin{bmatrix} v(t) \\ w(t)^{*} R(t) \\ F(t) \\ T(t) \end{bmatrix}$$