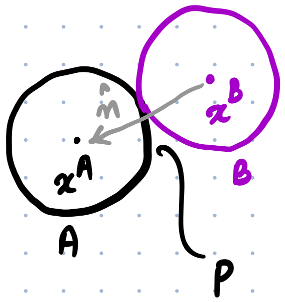
Rigid Body Collisions



Radius of sphere A: r^A
B: r^B

Mass " " A: m^A
B: m^B

Velocity " " A: v^A
B: v^B

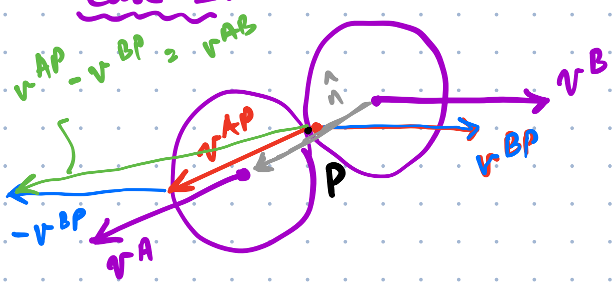
* There are no rotational effects.

Velocity of collision point P for sphere A: $v^{AP} = v^A$
B: $v^{BP} = v^B$

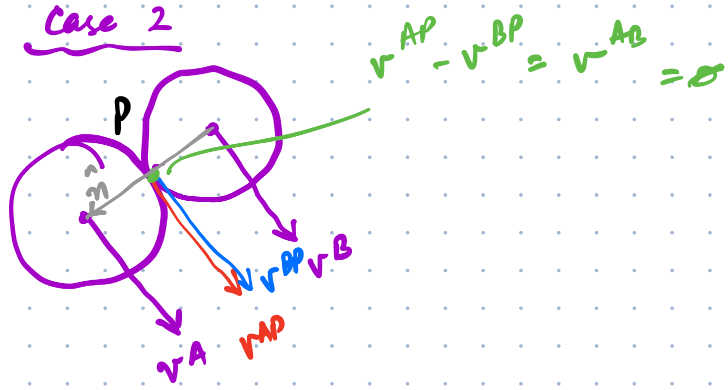
Collision detection ?

1. compute normal $\hat{n} = (p^A - p^B)/|p^A - p^B|$ *
2. Compute relative velocity at P: $v^{AB} = v^{AP} - v^{BP}$

Case 1.



Case 2



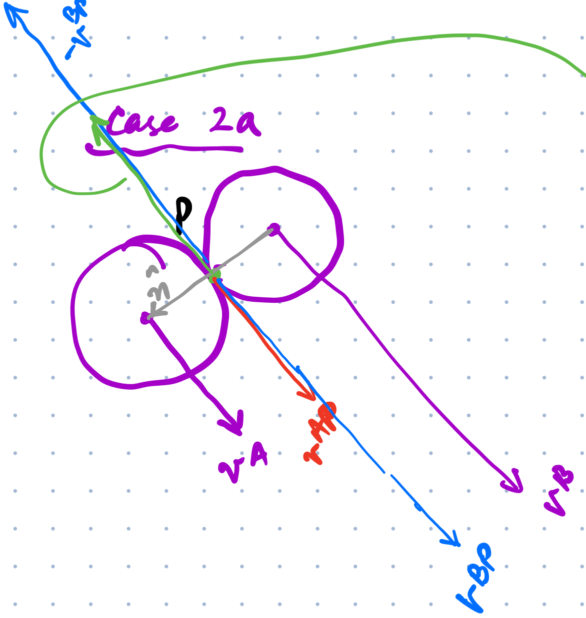
Case 3

Case 2 $v^{AB} \cdot \hat{n} = \emptyset$ Resting contact

Case 1 $v^{AB} \cdot \hat{n} > \emptyset$ Bodies are moving away from each other

Case 3 $v^{AB} \cdot \hat{n} < \emptyset$ Imminent Collision.

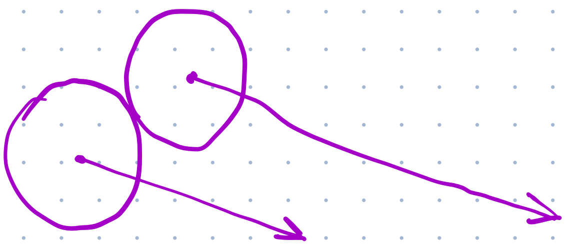
Case 2a



$v^{AP} - v^{BP} = v^{AB}$

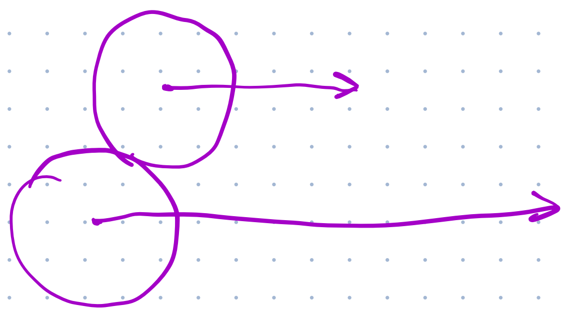
$v^{AB} \cdot \hat{n} = 0 \quad \because \quad v^{AB} \perp \hat{n}$

Case 4



no collision

Case 5



collision

① Newton's Law of Restitution for Instantaneous Collision with No Friction.

impulse: an infinite force applied for a very short duration.

impulse is equal to the change in momentum

$J = \Delta P$ ⇝ P represents momentum

impulse

$= mv_1 - mv_2$

after

velocity before the impulse was applied

$$v_2 = v_1 - \frac{J}{m}$$

- no gravity
- no friction
- conservation of momentum : J for the first body is equal to $-J$ of the second body.

② Empirical model of frictionless collisions

$$\underset{=}{v_2^{AB} \cdot n} = -e\, \underset{=}{v_1^{AB} \cdot n}$$

Relative velocity of spheres A and B after collision.

Relative velocity of spheres A and B before collisions

e is called the co-efficient of restitution.

- $e = 1$, elastic collision, no loss of K.E.
- $e = \emptyset$, perfectly inelastic collision, total loss of K.E.
- $\emptyset < e < 1$, some loss of K.E.

GOAL: given ① and ②, we want to solve for the velocities of spheres A and B after collision.

v_1^{AP} : velocity of P in A before collision

v_1^{BP} : " B "

require

$v_2^{AP} = ?$

$v_2^{BP} = ?$

$$(A) \text{ — } \quad v_2^{AP} = v_1^{AP} + \frac{jn}{m^A}$$

$$v_2 = v_1 - \frac{J}{m}$$

$$(B) \text{ — } \quad v_2^{BP} = v_1^{BP} - \frac{jn}{m^B}$$

Law of conservation of momentum holds

$v_1^{AP}, v_2^{AP}, v_1^{BP}, v_2^{BP}, n \in \mathbb{R}^d \quad (d=2,3)$

$j, m^A, m^B \in \mathbb{R}$

impulse j acts along $\hat{n}$

Abuse of notation: $n = \hat{n}$

Subtract (B) from (A)

$$\left(v_2^{AP} - v_2^{BP}\right) = \left(v_1^{AP} - v_1^{BP}\right) + \left(\frac{jn}{m^A} + \frac{jn}{m^B}\right)$$

$$\Rightarrow \quad \underset{?}{v_2^{AB}} = v_1^{AB} + \left(\frac{1}{m_A} + \frac{1}{m_B}\right) \overset{?}{j} n \quad \text{—— (C)}$$

We have from (2)

$$v_2^{AB} \cdot n = -e \, v_1^{AB} \cdot n \quad \text{—— (D)}$$

$$\left\{ v_1^{AB} + \left(\frac{1}{m^A} + \frac{1}{m^B}\right) \underset{?}{j} n \right\} \cdot n = -e \, v_1^{AB} \cdot n$$

$$\Rightarrow \quad \underbrace{v_1^{AB} \cdot n}_{\in \mathbb{R}} + \left(\frac{1}{m^A} + \frac{1}{m^B}\right) j \underbrace{n \cdot n}_{1} = -e \, \underbrace{v_1^{AB} \cdot n}_{\in \mathbb{R}}$$

$$\Rightarrow \quad j = -\left\{(1+e) \, v_1^{AB} \cdot n\right\} \left(\frac{1}{m^A} + \frac{1}{m^B}\right)^{-1}$$

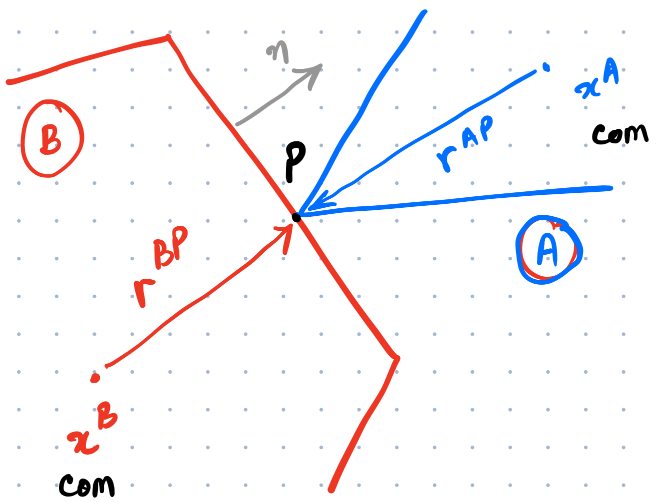
For A. What is j for B. $-j$

$$v_2^A = v_1^A + \frac{jn}{m^A}$$

$$v_2^B = v_1^B - \frac{jn}{m^B}$$

Collision Response.

Rigid Body Collisions with Rotational Effects



Collision point P in A : r^{AP}

B : r^{BP}

Velocity of P in A : $v_1^A + \omega_1^A \times r^{AP} = v_1^{AP}$

B : $v_1^B + \omega_1^B \times r^{BP} = v_1^{BP}$

angular velocity

linear velocity

We are interested in v_2^A, v_2^B, ω_2^A and ω_2^B

post-collision velocities

Mass of A : m^A

B : m^B

Inertia tensor of A : I^A (world)

B : I^B

Previously,

$$v_2^A = v_1^A + jn/m^A$$

$$\omega_2^A = \underbrace{\omega_1^A}_{\in\mathbb{R}^3} + \underbrace{\underbrace{(I^A)^{-1}}_{\in\mathbb{R}^{3\times3}}\ \underbrace{r^{AP}\times jn}_{\in\mathbb{R}^3}}_{\in\mathbb{R}^3}$$

Similarly,

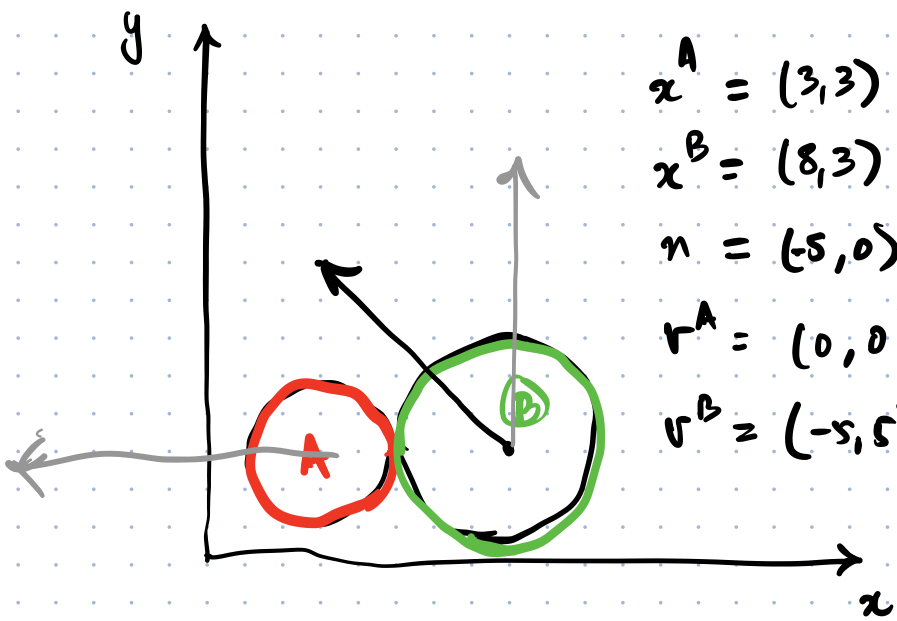
$$v_2^B = v_2^B - jn/m^B$$

$$\omega_2^B = \omega_1^B - (I^B)^{-1}\, r^{BP}\times jn$$

where

$$j = \frac{-(1+e)\, v^{AB}\cdot n}{\left(\frac{1}{m^A}+\frac{1}{m^B}\right) + \underbrace{n\cdot (I^A)^{-1}(r^{AP}\times n)\times r^{AP} + n\cdot (I^B)^{-1}(r^{BP}\times n)\times r^{BP}}_{\text{Rotational Effects}}}$$

EXERCISE 1



$x^A = (3,3)$

$x^B = (8,3)$

$n = (-5,0) = (-1,0)$

$r^A = (0,0)$

$r^B = (-5,5)$

$r^{AB} = (0,0) - (-5,5)$

$= (5,-5)$

$v^{AB} \cdot n = (5)(-1) + (-5)(0) = -5 < 0$

Kaboom

let $m^A = 1$, $m^B = 1$, $e = 1$.

$$\boxed{\begin{aligned} v_2^A &= v_1^A + \frac{jn}{m^A} \\ v_2^B &= v_1^B - \frac{jn}{m^B} \end{aligned}}$$

? ✓ $j = ?$

$v_1^A = (0,0)$

$v_1^B = (-5,5)$

$n = (-1,0)$

$m^A = 1$ *

$m^B = 1$ *

$j = ?$

$$j = -\{(1+e)\, v_1^{AB} \cdot n\}\left(\frac{1}{m^A} + \frac{1}{m^B}\right)^{-1}$$

For A. What is j for B. $-j$

$e = 1$ *

$v_1^{AB} = (5,-5)$

$$= -(1+1)\left[(5,-5)\cdot(-1,0)\right]\left(\frac{1}{1} + \frac{1}{1}\right)^{-1}$$

$$= \frac{-(2)(-5)}{2}$$

$$= 5$$

$$v_2^A = v_1^A + \frac{j n}{m^A}$$

$$= (0,0) + \frac{5(-1,0)}{1}$$

$$= (0,0) + (-5,0)$$

$$= (-5,0)$$

$$v_2^B = v_1^B - \frac{j n}{m^B}$$

$$= (-5,5) - \frac{5(-1,0)}{1}$$

$$= (-5,5) + (5,0)$$

$$= (0,5)$$

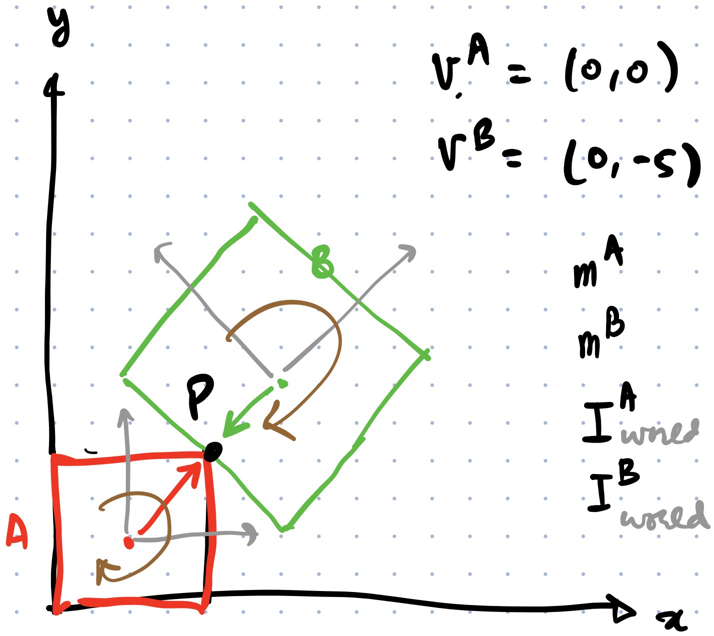
EXERCISE 2

$x^A = (2,2) = (2,2,0)$ ✓

$x^B = (6,6) = (6,6,0)$

$v^A = (0,0)$, $\omega^A = (0,0,0)$

$v^B = (0,-5)$, $\omega^B = (0,0,0)$



m^A

m^B

$\left.\begin{matrix} I^A_{world} \\ I^B_{world} \end{matrix}\right\} \begin{matrix} R^A \\ R^B \end{matrix}$