Bouncing ball

Collision detection

1. Approximate time to collide $t_c = \frac{x}{v}$
2. Set $x = 0$ and $v = -v(t + t_c)$
   - Flip v to indicate that the ball is now going back up again

Problem

Bouncing ball

Collision detection

1. Approximate time to collide $t_c = \frac{x}{v}$
2. Set $x = 0$ and $v = -v(t + t_c)$
   - Flip v to indicate that the ball is now going back up again

Problem

v is larger than had we calculated t_c exactly right (that's because the particle is under constant acceleration). Consequently energy is not conserved.

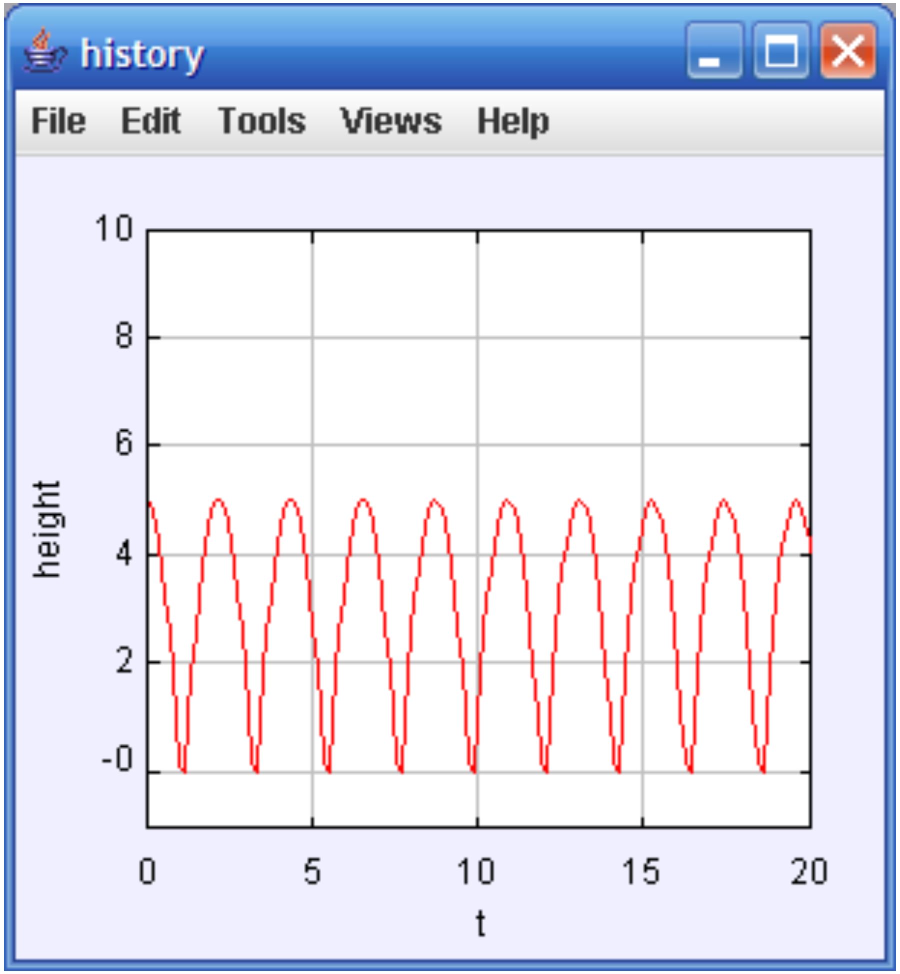
Bouncing ball

- Use the Law of Conservation of Energy to compute the velocity of the ball when it touches ground.
- The ball was released at height h. We know the total energy of the system, which is mgh. At the start the kinetic energy is 0.
- When the ball touches the ground, its potential energy reduces to 0. Since the total energy remains the same, all of its energy is now kinetic energy.

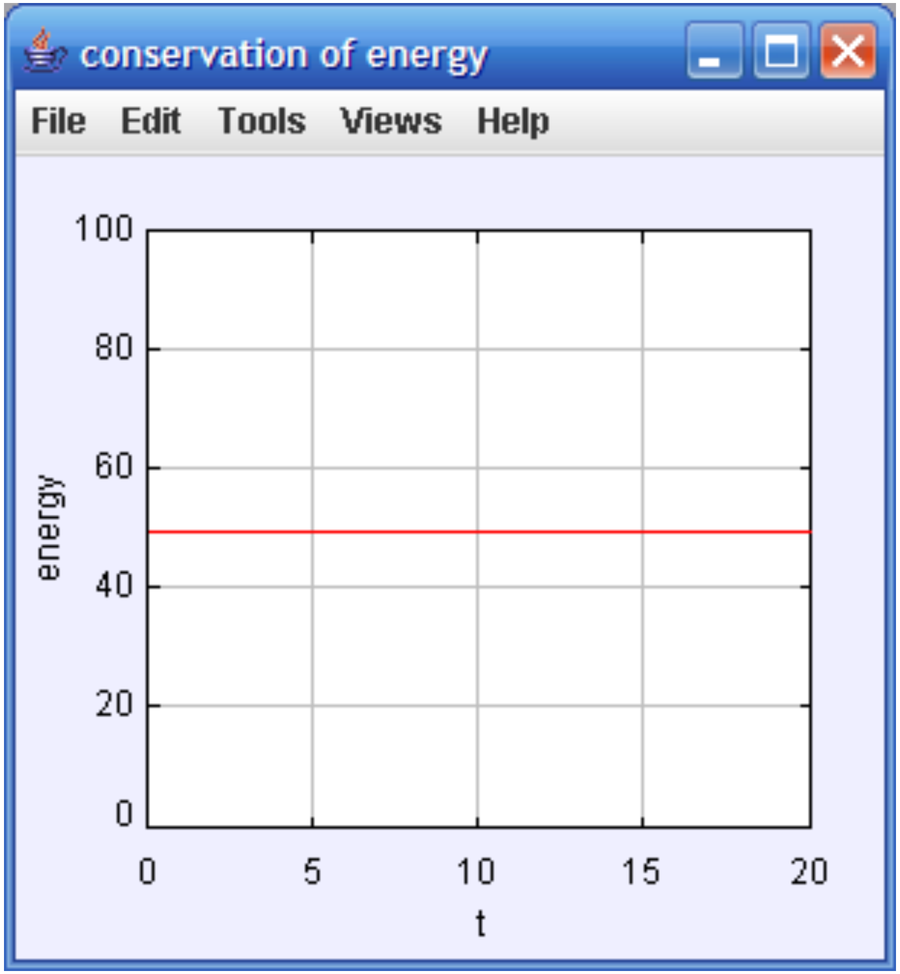
$$\frac{1}{2}mv^2 = mgh$$
$$v = \sqrt{2gh}$$

i.e., set $x = 0$ and $v = -\sqrt{2gh}$ at collision time.

Bouncing ball



Ball doesn't enter the floor



Energy is conserved

2D elastic band

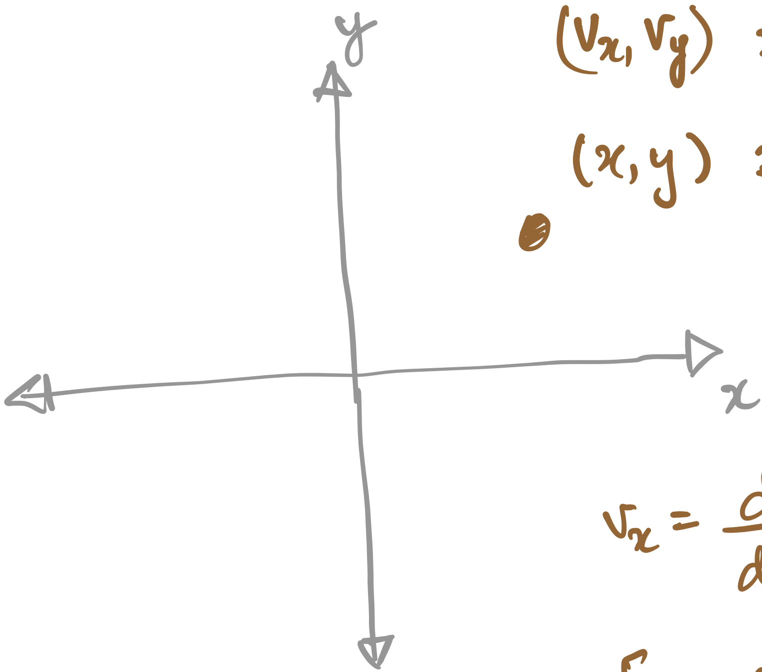
Simulate a ball (point mass) attached to the origin via an elastic band (or a *spring sitting in a plane*).

- We assume that the rest length of the band is 0.
- Hook's law describes the relationship between the extension of the band and the force it applies on the attached ball

Hook's law in 1D

$$F = -kx,$$

where x is the displacement from the rest length (in this case 0), and k is the spring constant for the elastic band. F is the force on the ball.



(v_x, v_y) : velocity

(x, y) : position

$$v_x = \frac{dx}{dt}$$

$$F_x = m\frac{dv_x}{dt}$$

$$v_y = \frac{dy}{dt}$$

$$F_y = m\frac{dy}{dt}$$

v_x, v_y, x, y, t

state variables

2D elastic band

Option 1

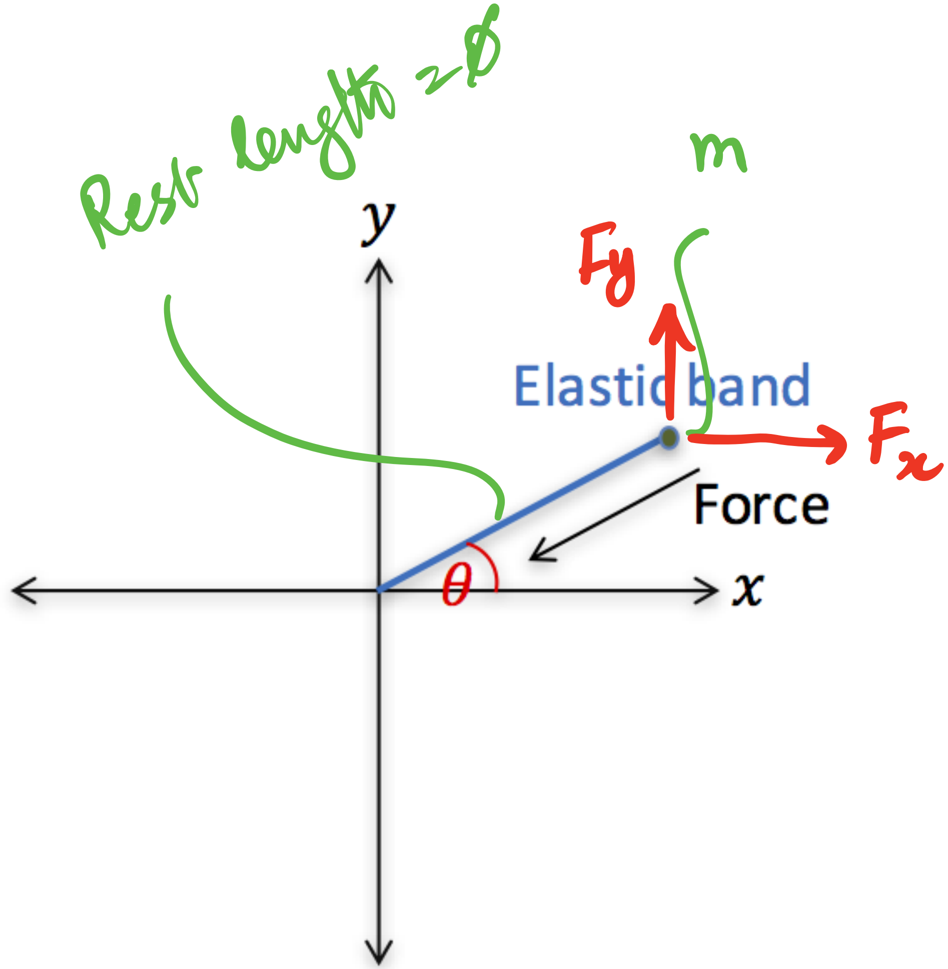
- Use Hook's Law in 2D

$$F = -k \underbrace{\left(\sqrt{x^2 + y^2}\right)}_{\text{displacement}}$$

So

$$F_x = -k \left(\sqrt{x^2 + y^2}\right) \cos(\theta)$$

$$F_y = -k \left(\sqrt{x^2 + y^2}\right) \sin(\theta)$$



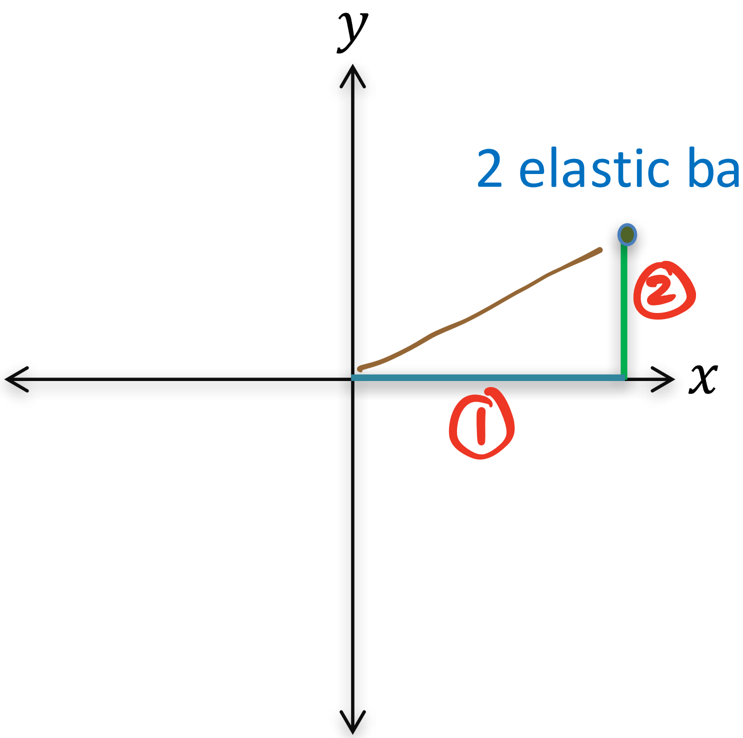
2D elastic band

Option 2

Replace 1 2D elastic band with 2 1D elastic bands. The first band sits along the x-axis; whereas, the second band sits along the y-axis.

$$F_x = -kx$$

$$F_y = -ky$$



2D elastic band

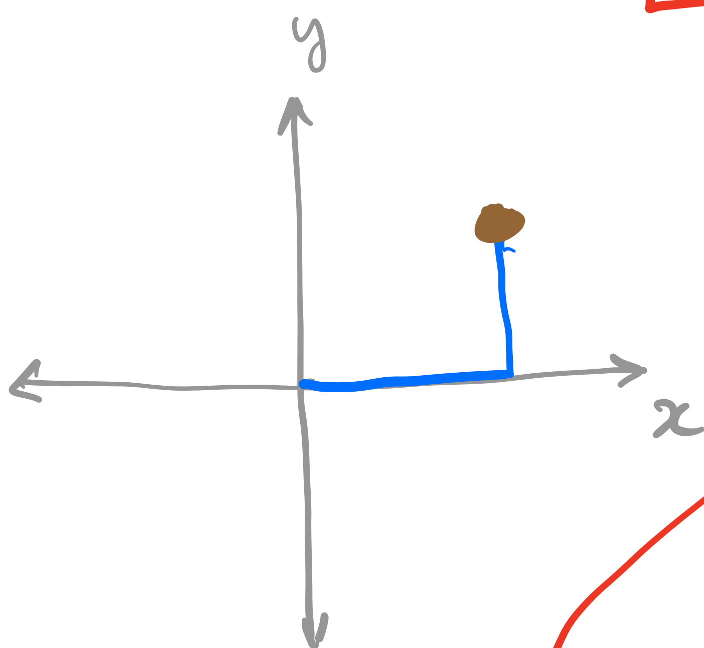
Add a damping force that is proportional to the velocity of the ball

Model

damping

$$F_x = -kx - cv_x$$
$$F_y = -ky - cv_y$$

$$v_x = \frac{dx}{dt}$$
$$v_y = \frac{dy}{dt}$$
$$F_x = m\frac{dv_x}{dt}$$
$$F_y = m\frac{dv_y}{dt}$$



Hook's law

Friction.

Acts against velocity

2D elastic band

Add a damping force that is proportional to the velocity of the ball

Model

$$F_x = -kx - cv_x$$
$$F_y = -ky - cv_y \qquad \underbrace{-mg}_{\text{gravity}}$$

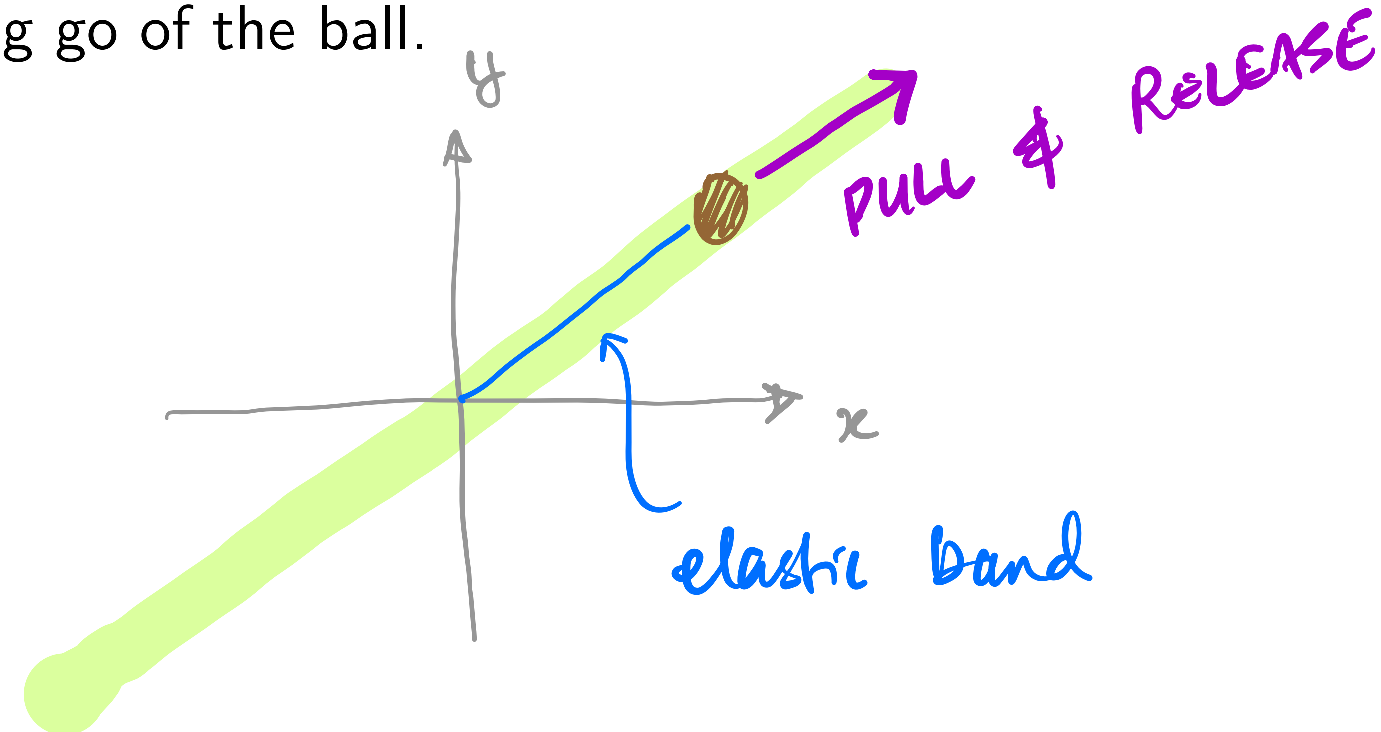
How many state variables?
5 = 2 for positions, 2 for velocity and 1 for time

Notice that we are consider t as a state variable as well. This is not exactly right, but it makes for easier implmentations.

2D elastic band

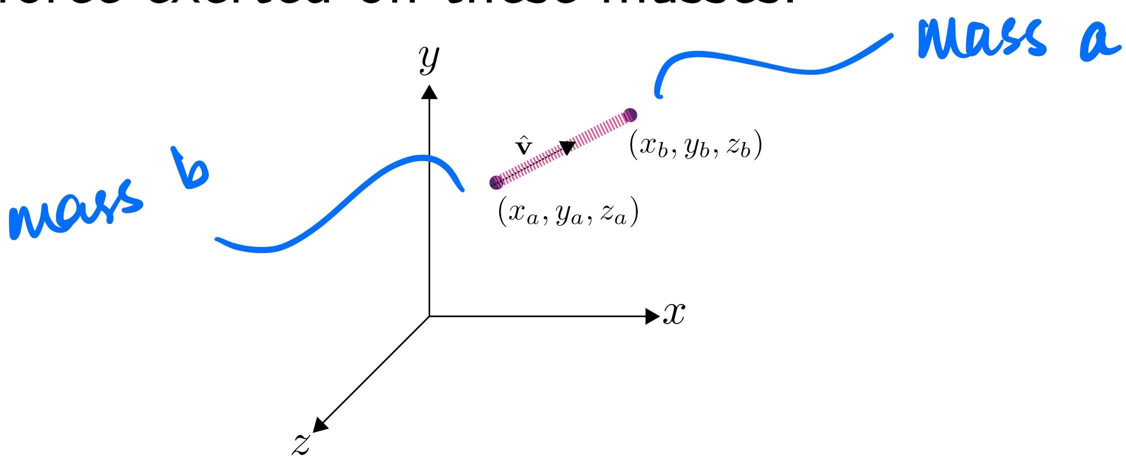
Interaction

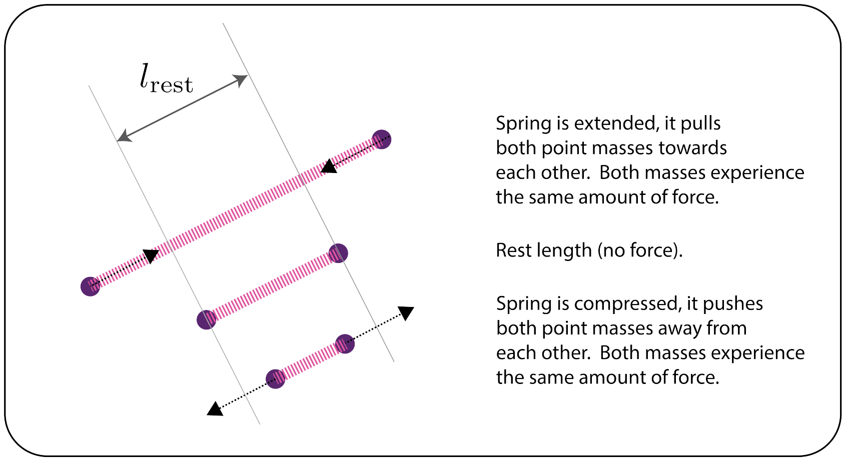
- User interacts with the ball by dragging it to a new location.
- Dragging to the new location changes the x and y extensions of the elastic band, effectively changing the forces acting on the ball.
- This is similar to grasping a real ball attached to a spring, and then letting go of the ball.



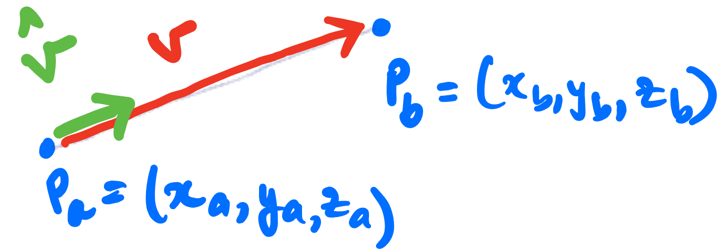
Mass-Spring systems in 3D

Consider a spring with rest length l and spring constant k. The spring is connected at two point masses located at $\mathbf{p}_a = (x_a, y_a, z_a)$ and $\mathbf{p}_b = (x_b, y_b, z_b)$, respectively. Our goal is to estimate the spring force exerted on these masses.





Spring Deformation and Axis



Spring axis vector:

$$\mathbf{v} = (x_b - x_a, y_b - y_a, z_b - z_a)$$

Current length of the spring:

$$l_{\text{current}} = \|\mathbf{v}\| = \underbrace{\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2}}$$

Deformation:

$$d = l_{\text{current}} - l_{\text{rest}} \quad \begin{cases} + & \text{when expanded} \\ - & \text{when compressed} \end{cases}$$

The deformation is positive if the spring is extended, and it is negative if the spring is compressed.

Hook's Law

Unit-vector along spring axis:

$$\hat{\mathbf{v}} = \frac{1}{l_{\text{current}}}(x_b - x_a, y_b - y_a, z_b - z_a)$$

Use the unit vector $\hat{\mathbf{v}}$ to compute force direction. Recall that this vector points towards $\mathbf{p}_b$. If the spring is extended, the mass at location $\mathbf{p}_b$ will experience a force in the direction of the mass at location $\mathbf{p}_a$. Therefore, the spring exerts the following force on the mass at location $\mathbf{p}$:

$$f_{\text{on point mass at } \mathbf{p}_b} = -kd\hat{\mathbf{v}}$$

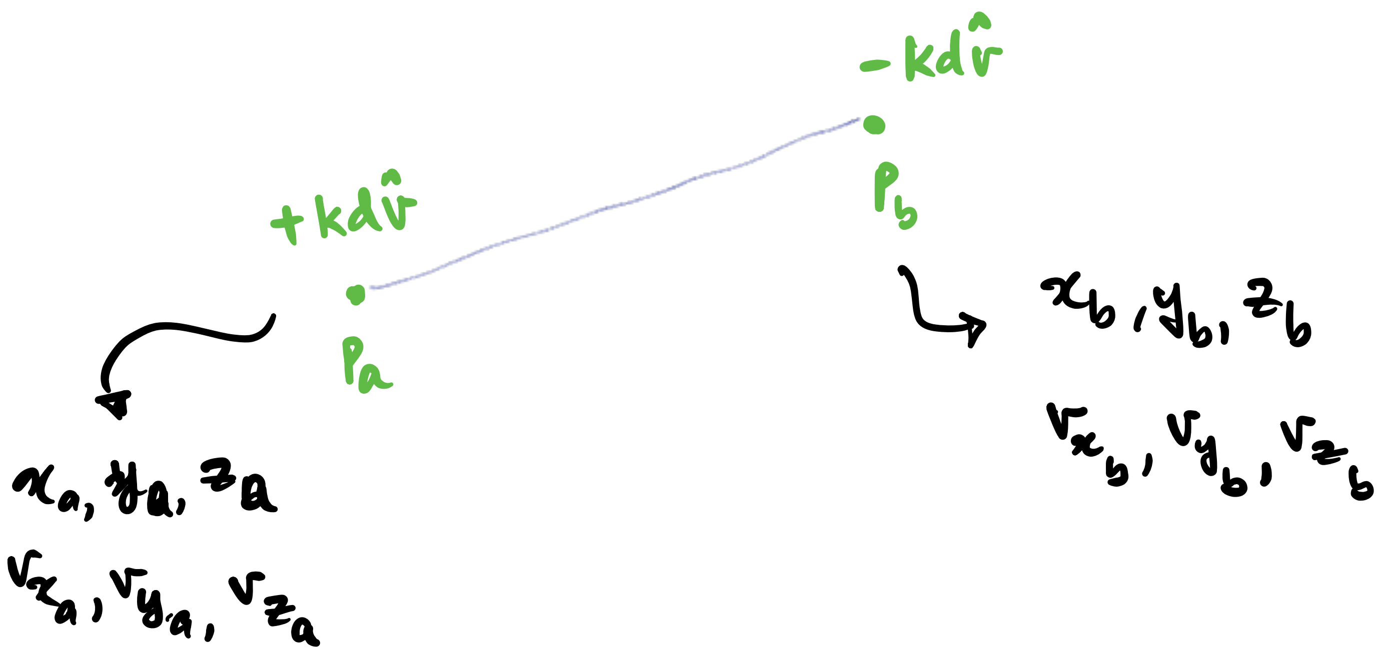
This expression also works when the spring is compressed. When the spring is compressed, d is negative. Therefore, the force on mass at $\mathbf{p}_b$ is along $\hat{\mathbf{v}}$.

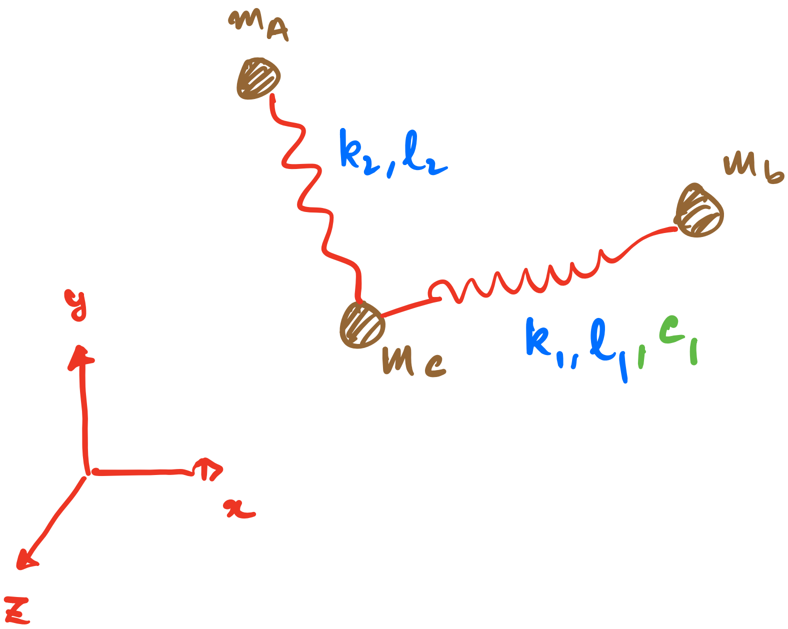
Hook's Law

Similarly, the force on the mass at location $\mathbf{p}_a$ is

$$f_{\text{on point mass at }\mathbf{p}_a} = +kd\hat{\mathbf{v}}.$$

This is just the opposite of the force on mass at $\mathbf{p}$





mA
k2, l2
mb
mc
k1, l1, c1
y
x
z

Projectile motion

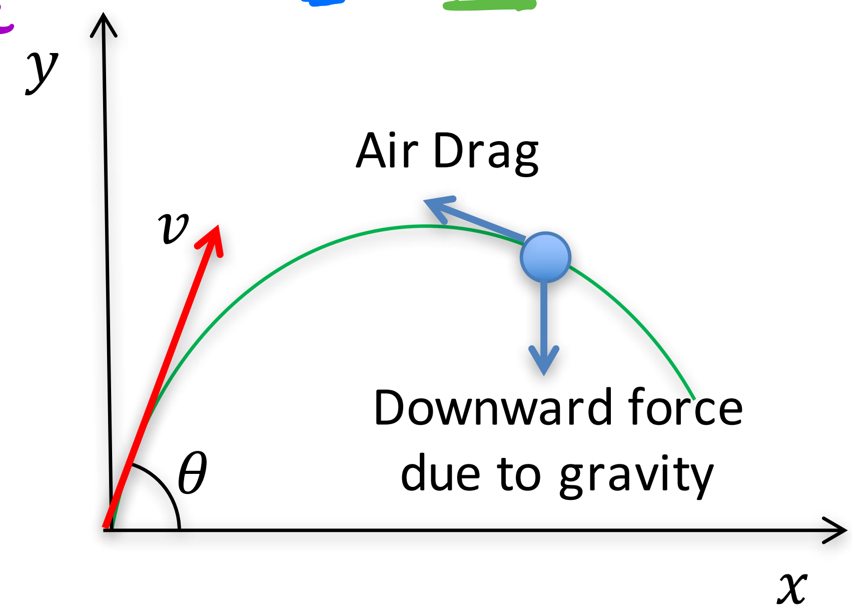
$$F = ma = mx''$$

FORCES

$$v = \frac{dx}{dt} = x'$$

$$mx'' = -\gamma x'$$
$$my'' = -\gamma y' - mg$$

Here γ is the friction constant, m is the mass of the particle, and g is the acceleration due to gravity.
This model doesn't take into account the effects of earth's gravitational field.



Force : F_y

: F_x

Projectile motion

Gravitational force between two masses M and m with is given by Newton's Law of Universal Gravitation

$$F = \frac{GM}{(R+y)^2},$$

where $R + y$ is the distance between their centres. G is the gravitational constant.

$$G = 6.674 \times 10^{-11} N(m/kg)^2$$

The value of g is merely a simplification given by

$$g = \frac{GM}{R^2}.$$

Projectile motion

To model a projectile near the surface of the earth we use

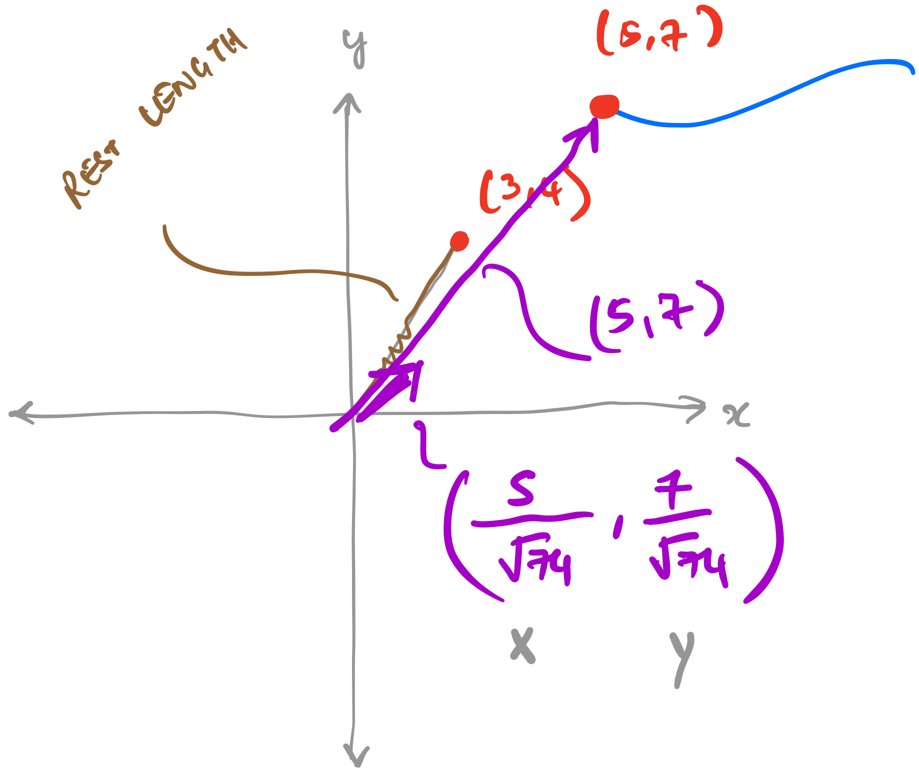
$$mx'' = -\gamma x'$$

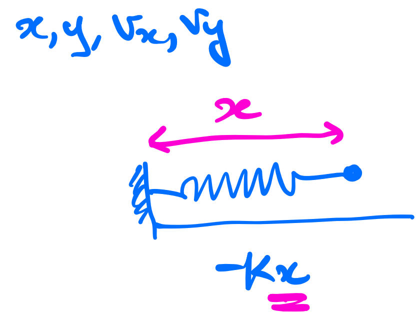
$$my'' = -\gamma y' - \frac{GM}{(R+y)^2}$$

FORCES

Unlike previous models that you have seen in this course, the above equations have no *analytical solution*. You'll have to solve them numerically.

State variables: x, y, v_x, v_y





REST LENGTH = $\sqrt{3^2+4^2} = 5$

CURRENT LENGTH = $\sqrt{5^2+7^2} = \sqrt{25+49} = \sqrt{74}$

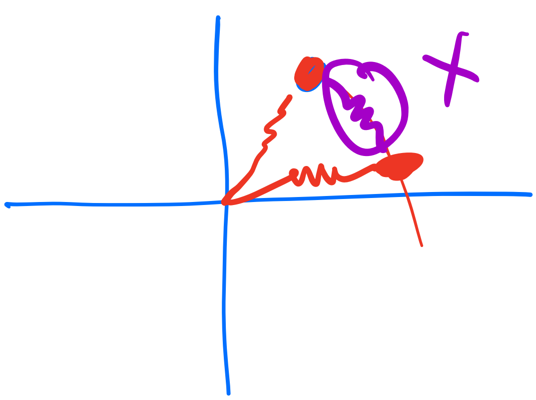
DEFORMATION = $(\sqrt{74} - 5)$

$$F_x = -\boxed{k}\,(\sqrt{74} - 5)\left(\frac{5}{\sqrt{74}}\right)$$

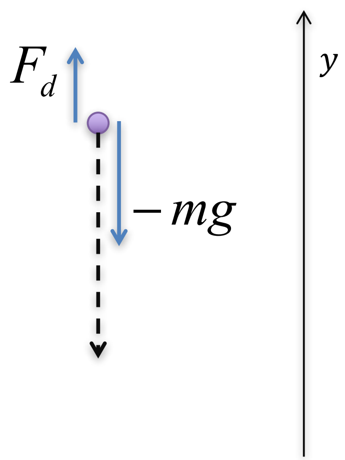
$$F_y = -\boxed{k}\,\underbrace{(\sqrt{74} - 5)}_{\text{Deformation}}\underbrace{\left(\frac{7}{\sqrt{74}}\right)}_{x/y}$$

dir

spring constant



Free-falling particle



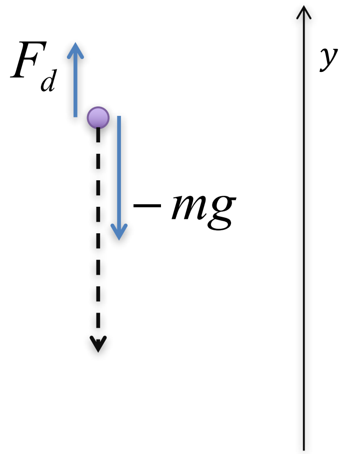
- Force acting on a particle of mass m falling under gravity is

$$F = -mg + F_d,$$

where F_d is the drag force experienced by the particle as it moves through the air.

- F_d is a velocity dependent drag force. It increases with velocity and at some point, it will become equal to the mg, i.e., $F_d = mg$. The velocity at which this occurs is referred to the terminal velocity of the particle.

Free-falling particle



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  where F_d is the drag force experienced by the particle as it moves through the air.
- F_d is a velocity dependent drag force. It increases with velocity and at some point, it will become equal to the mg, i.e., $F_d = mg$. The velocity at which this occurs is referred to the terminal velocity of the particle.
- Once terminal velocity is achieved the particle experiences 0 net force. The particle still continues to fall at a constant velocity. Why is that?

Free-falling particle

Terminal velocity

The velocity at which the motion of an object through a fluid is constant due to the drag force exerted by that fluid.
Terminal velocity depends upon both the particle and the medium through which it is moving.

Example: Falling pebble

Consider the fall of a pebble of mass $10^{-2}\ kg$. The terminal velocity of this pebble is $30\ m/s$.

- How long with it take for this pebble to achieve terminal velocity?
- How much distance will this pebble cover before it achieves terminal velocity?

Observation: The pebble will cover around $50\ m$ to achieve the terminal velocity. This will take around $3\ s$.

So if we are dealing with a pebble simulation across these distances (or times), we need to take into account terminal velocity.

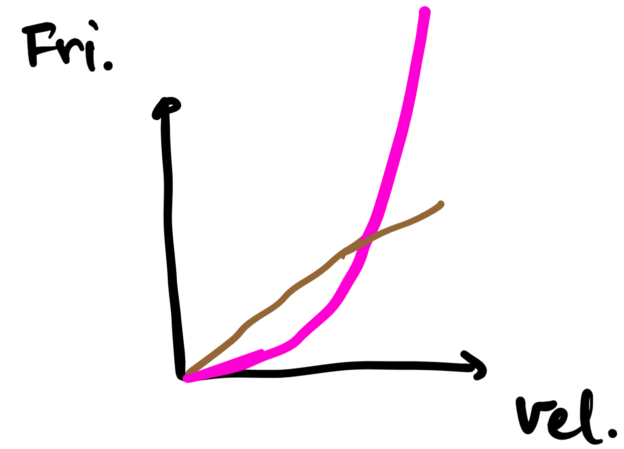
Example: Falling pebble

Takeaway: even when modeling simple systems, such as a free falling particle, we need to carefully evaluate the conditions so as not to miss important effects.

Describing drag in terms of terminal velocity

- Linear drag

$$F_{1,d} = C_1 v = mg\frac{v}{v_{1,t}}$$



- Quadratic drag

$$F_{2,d} = C_2 v^2 = mg\left(\frac{v}{v_{1,t}}\right)^2$$

Modeling a falling coffee filter

Sketch

- Observe a falling coffee filter and record positions vs. times.
- Estimate velocities and accelerations via finite differences.
- Estimate terminal velocity. Recall that the object falls with constant velocity once terminal velocity is achieved.
- Identify the relationship between acceleration and velocity. Is it linear or quadratic?
- Right down the equations taking into account your findings.
- Run the simulation and see if it matches your observations.

Modeling a falling coffee filter

- Observe a falling coffee filter and record positions vs. times.
- Estimate velocities and accelerations via finite differences.
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- Identify the relationship between acceleration and velocity. Is it linear or quadratic?
- Right down the equations taking into account your findings.
- Run the simulation and see if it matches your observations.

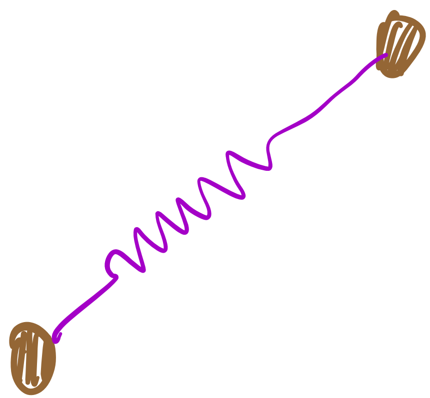
```
//Falling coffee filter
//Time ( s )    Position ( m )
0.2055  0.4188
0.2302  0.4164
0.255   0.4128
0.2797  0.4082
0.3045  0.4026
0.3292  0.3958
0.3539  0.3878
0.3786  0.3802
0.4033  0.3708
0.428   0.3609
0.4526  0.3505
0.4773  0.34
0.502   0.3297
0.5266  0.3181
0.5513  0.3051
0.5759  0.2913
0.6005  0.2788
0.6252  0.2667
0.6498  0.2497
0.6744  0.2337
0.699   0.2175
0.7236  0.2008
0.7482  0.1846
0.7728  0.1696
0.7974  0.1566
0.822   0.1393
0.8466  0.1263
```

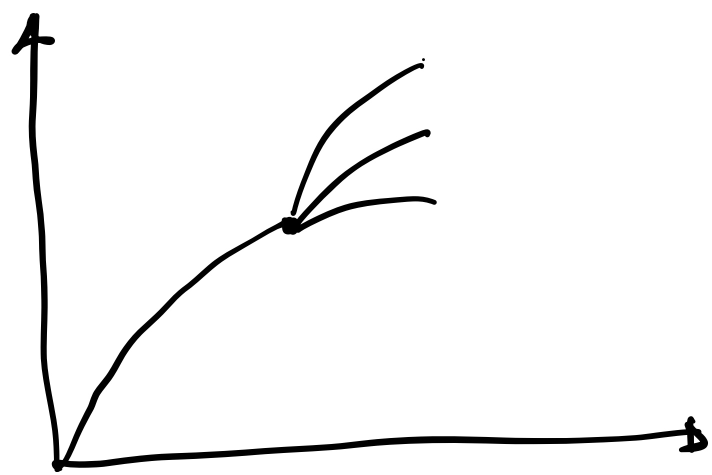
Modeling a falling coffee filter

Takeaway: it is sometimes possible to infer dynamics from empirical data

Simulating multiple objects

- So far we have simulated single objects
- Now we discuss how to simulate multiple objects?
  - The number of objects is a parameter for the simulation.

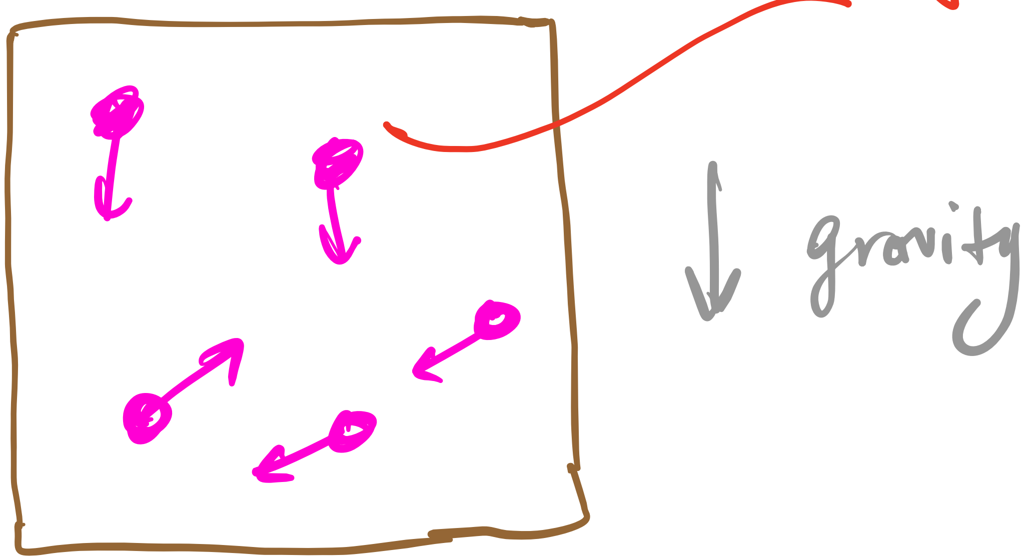




Simulating a collection of balls in a square

- Balls move in 2D
- Random initial positions and velocities
- Balls move under the influence of gravity
- Balls bounce off the walls
- Balls pass through each other (i.e., no collisions between balls)
- No friction

x, y, v_x, v_y in 2D



20 eq.

Simulating a collection of balls in a square

Question 1

Say we are interested in simulating n balls in a square. What is the state size of our simulation?

Question 2

How do we set up the initial state for our simulation, i.e., the initial locations and initial velocities for each ball?

REJECTION SAMPLING.

Simulating a collection of balls in a square

Question 1

Say we are interested in simulating n balls in a square. What is the state size of our simulation?

Answer 1

$4n$, (x, y) locations and (v_x, v_y) velocities for each ball.

Question 2

How do we set up the initial state for our simulation, i.e., the initial locations and initial velocities for each ball?

Simulating a collection of balls in a square

Question 1

Say we are interested in simulating n balls in a square. What is the state size of our simulation?

Answer 1

$4n$, (x, y) locations and (v_x, v_y) velocities for each ball.

Question 2

How do we set up the initial state for our simulation, i.e., the initial locations and initial velocities for each ball?

Answer 2

Random positions and velocities.

Simulating a collection of balls in a square

What are we missing in this simulation?

- Not handling collisions between balls
  - If only a few balls in a very large square, ball-ball collisions may be rare event.
  - If a lot of balls crammed in a small space, we can't really ignore ball-ball collisions.

Ball-ball collisions

- Ball-ball collisions are difficult to do efficiently.
  - Unlike ball-wall collisions, where only one object is moving, in ball-ball collision, both objects are moving.

Naive approach

- At each time step, inspect each pair for possible collision.
- For n balls this leads to n^2 inspections.

Other things to consider

What if three balls collide with each other at the same instant?
What if n balls collide with each other at the same instant?

Object-object collisions

- Efficient collisions between multiple objects is very challenging
- Most simulations only consider these when absolutely necessary
  - Gas molecules are small, so when simulating low-density gases in large volumes, inter-molecules collisions are sometimes ignored.

What other things have we ignored in our simulation containing multiple balls in a square?

Brainstorm

What other things have we ignored in our simulation containing multiple balls in a square?

- Ball-wall collisions ignore the effects of impact on the wall (and the balls)
  - If the balls were ball bearings, and the walls were made of thin aluminum then each collision would dent the wall.
  - The walls will get bent out of shape over time.
  - How would you model walls that bends overtime? This require some very complicated physics, large computational power, and sophisticated numerical techniques.
- We also didn't model the color of the balls
  - This would be of interest if we are intrested in light bounces or heat transfer.

Guidlines

- We need carefully identify what really needs to be modeled and simulated.
- We can make simulations arbitrarily complex by considering more things.
  - This makes it harder to produce simulations.
  - Simulations will be less efficient.
  - Simulations might become less useful.
- We need to know where to draw the line.

Correctly determining the applications of the simulation is an important first step in getting the model right

Summary

- Input, output and state variables
- Differential equations are used to model the behavior of state variables
- Numerical solvers for solving differential equations
  - Good numerical solvers really only exist for degree 1 differential equations.
- Transform higher order differential equations to multiple first-order differential equations.
  - This introduces extra state variables

Summary

- Use of indirect means to determine whether or not our simulation is correct.
  - We used our knowledge of the law of conservation of energy to identify the problem with our simulation
- Interactions
- Projectile motion
  - Our first encounter with an ODE that has no *analytical* solution
- Drag
  - Terminal velocity
- First exposure to infering dynamics from empricial data

Summary

Simulating multiple objects

- How to model the system?
- How to manage state space?
- Performance
- Problem set up or initialization

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