Continuous Systems

Simulation and Modeling (CSCI 3010U)

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Continuous systems simulation

- Time is treated as a continuous variable that drives the simulation
- Model is based upon differential equations, which describe how systems evolves over time, and how it responds to changes in input variables
- In this course, we will mostly deal with Ordinary Differential Equations (ODE), though Partial Differential Equations are used in some cases

control how various variables evolve over time.

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- **Input variable** are not controlled by the simulation
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- **Output variable** are things that we observe
  - The speed of the vehicle in the car simulator example
- **State Variables** are controlled by the differential equations
  - The speed of the vehicle in the car simulator example
  - The location of the vehicle in the car simulator example

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- A multi-variable function F that describes a relationship between the derivatives of the dependent variable (taken with respect to the independent variable)

Putting it all together, we get

$$F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \cdots, \frac{d^{(n-1)}x}{dt^{(n-1)}}\right) = \frac{d^n x}{dt^n}$$

ODE: comments

- $\frac{dx}{dt}$ denotes derivative of x w.r.t. t

$$\frac{d}{dt}(3x^3) = 9x^2$$

$$\frac{d^2}{dt^2}(3x^3) = 18x$$

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Examples

- $mx'' = F$ ⟶ $m\frac{d^2x}{dt^2} = F$
- $x' + 32x'' + x''' = 0$
- $x' + 34x = 32$

$$\mathcal{F}\left(t, x, F, m\right) = \frac{d^2x}{dt^2}$$

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Examples

- $mx'' = F$ (order is 2)
- $x' + 32x'' + x''' = 0$
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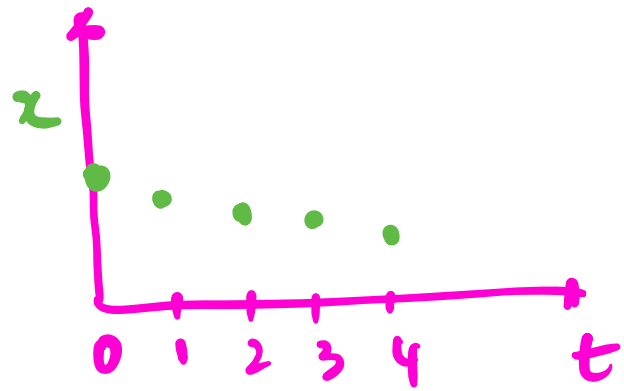
- $x' + 32x'' + x''' = 0$ (order is 3)
- $x' + 34x = 32$ (order is 1)

Solving differential equations

- Solution is the dependent variable $x = x(t)$ that satisfies the equation
- Key idea: integration

Example: $x'' = 2$ $\longrightarrow$ $\frac{d^2x}{dt^2} = 2$

Interest : $x(t) = ?$



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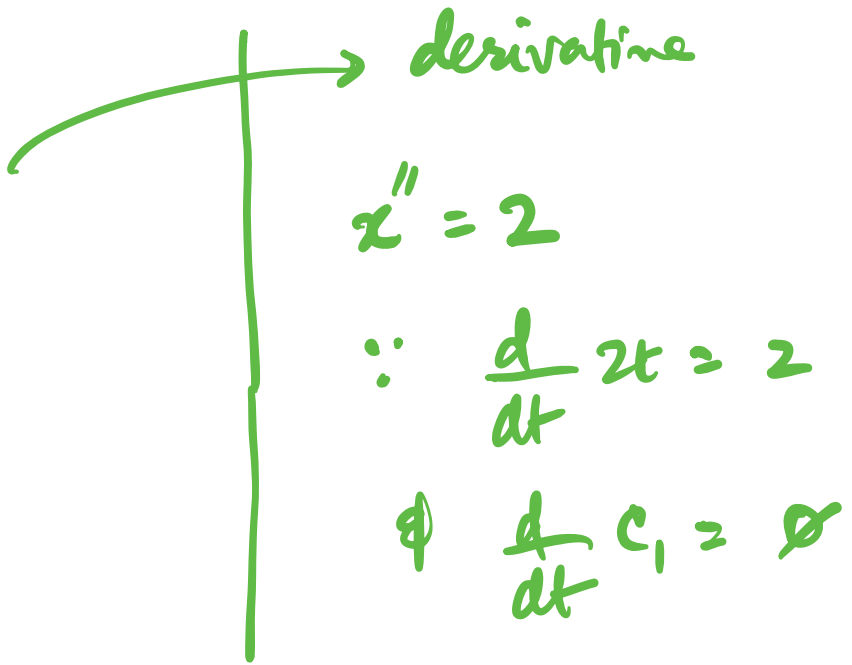

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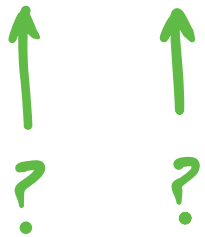

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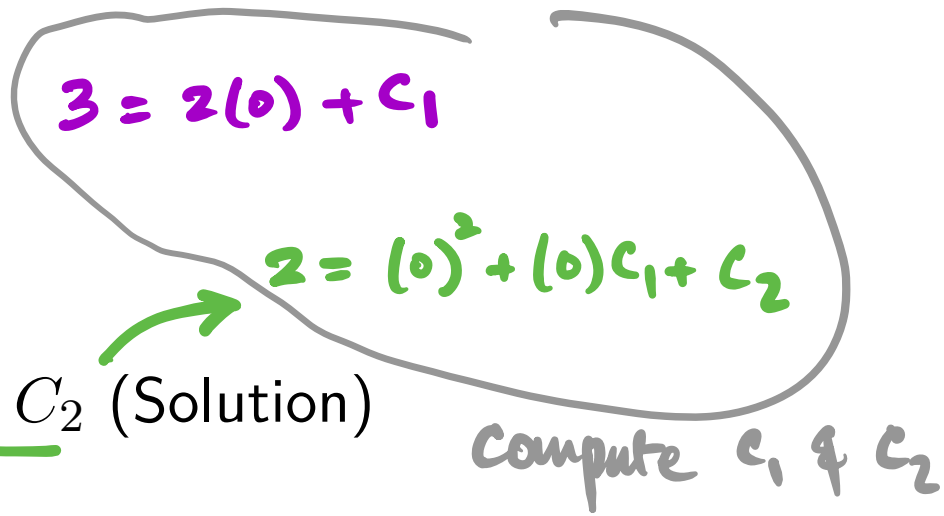

How to solve for C_1 and C_2, also called, constants of integration?

- Use initial or boundary conditions.
- Using initial conditions $x'(0) = 3$ and $x(0) = 2$, we get C_1=3 and $C_2 = 2$. The solution is $x(t) = t^2 + 3t + 2$.

1. $x' = 2t + c_1$

2. $x = t^2 + tc_1 + c_2$

Boundary Conditions: $x(0) = 2$, $x(10) = 7$

$2 = (0)^2 + (0)c_1 + c_2$

$7 = (10)^2 + (10)c_1 + c_2$

Aside: Constant of Integration

Notice that

$$\frac{df(x)}{dx} = 3x^2$$

for both when $f(x) = x^3$ or $f(x)x^3 + C$.

This suggests constant C disappears in the process of differentiation.

Therefore, when we integrate we add the constant C for the sake of completeness.

$$\int 3x^2 dx = 3\left(\frac{x^3}{3}\right) + C = x^3 + C$$

Furthermore, we will need other information to find the true value of C. Note also that there is nothing preventing $C = 0$.

Nth order ODEs

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- Boundary conditions
  - The values of $x(t)$ and its derivatives for two different values of t (if you have two co-efficients of integration)

Nth order ODE reducibility

Any explicit differential equation of order n

$$x^{(n)} = F\left(t, x, x', \cdots, x^{(n-1)}\right)$$

can be written as a system of n first-order differential equations by defining a new family of unknown functions

$$x^{(i-1)} = x_i$$

Notice the abuse of notation. Here $x^{(k)}$ denote the k-th derivative of x w.r.t. t.

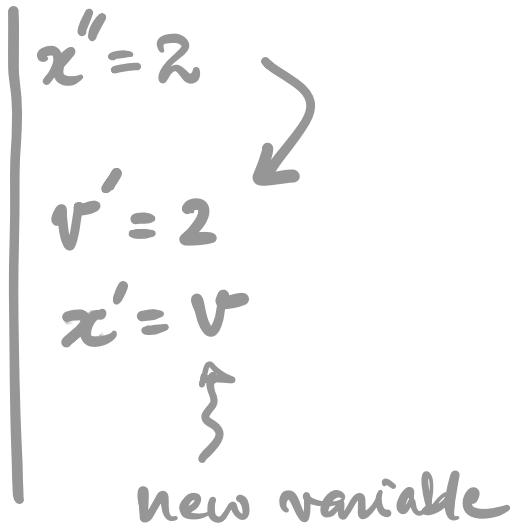
Nth order ODE reducibility

We can then represent the following n-th order ODE

$$x^{(n)} = F\left(t, x, x', \cdots, x^{(n-1)}\right)$$

with n first-order ODEs as follows

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{(n-1)}' &= x_n \\ x_n' &= F\left(t, x_1, x_2, \cdots, x_n\right) \end{aligned}$$



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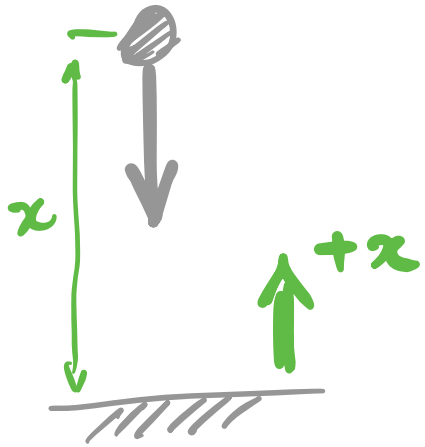
Example

The following 2nd order ODE

$$\frac{d^2x}{dt^2} = -g$$



$F = ma$

$\Rightarrow a = F/m$

$\Rightarrow a = -g$

$\Rightarrow \frac{d^2x}{dt^2} = -g$

can be reduced to

add a new variable v →

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -g$$

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We introduced a new variable v.

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  - E.g., the equation of motion for particle is modelled by a second order differential equation
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- Advantages of first order ODEs
  - First order equations are much easier to solve numerically
  - Very few numerical solvers available for higher order equations

Equation of motion

Newton's second law of motion:

> *"The acceleration of an object as produced by a net force is directly proportional to the magnitude of the net force, in the same direction as the net force, and inversely proportional to the mass of the object."*

Mathematically:

$a \propto F$ and $a \propto \frac{1}{m}$, and combining the two we get $F = ma$. Recall $a = \frac{d^2x}{dt^2}$, so $F = ma$ is a second order equation.

$$F = ma$$

$$\implies F = m\frac{d^2x}{dt^2}$$

Equation of motion

We introduce a new variable velocity $v = \frac{dx}{dt}$, and get the following first order system of equations

$$v = \frac{dx}{dt}$$
$$F = m\frac{dv}{dt}$$

Solving ODEs numerically

General idea: given a solution $x(t)$ at time $t = t_0$, incrementally step forward in time to find $x(t + \Delta t)$

Example: lets consider the equation of motion

$$F = m\frac{dv}{dt} \implies \Delta v = \left(\frac{F}{m}\right)(\Delta t)$$

$$v = \frac{dx}{dt} \implies \Delta x = (v)(\Delta t)$$

update rules

We can use Δv and Δx to update the **current** values of v and x, respectively.

state variables.

Do need the initial state.

Solving ODEs numerically: an example

What is the value of x at time $t = 3$?

Model

$$v(t + \Delta t) = v(t) + \Delta v = v(t) + (F/m)(\Delta t)$$
$$x(t + \Delta t) = x(t) + \Delta x = x(t) + (v)(\Delta t)$$

Setup

- $m = 1$, $F = 1$ (other quantities used in the simulation)
- $v(0) = 0$, $x(0) = 0$ (initial state), $\Delta t = 1$ (time step)

Solution

- $v(1) =$
  - $x(1) =$
- $v(2) =$
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- $v(3) = 3$
  - $x(3) = 6$

Solving ODEs numerically: practical considerations

- Have to choose Δt carefully
  - Δt too small; the simulation can become very slow
  - Δt too large; the simulation can become very inaccurate
  - Advanced techniques can change Δt when solving equations to maintain acceptable accuracy and speed

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- Often times Δt used for solving ODEs is much smaller than the one used to update the display

Choice of ΔT

- Molecular activity
- Evolution of an ecosystem
- Galaxy formation

Choice of ΔT

- Molecular activity (fraction of a millisecond)
- Evolution of an ecosystem
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Choice of ΔT

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- Molecular activity (fraction of a millisecond)
- Evolution of an ecosystem (months or years)
- Galaxy formation (millions or billions of years)

Simulation loop

- Advance the simulation

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- Get the user response

Simulation loop

- Advance the simulation
- Display current results
- Get the user response
- Repeat

Displaying results

- Real-time or not?

Displaying results

- Real-time or not?
- Timescale

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- Response and interactivity

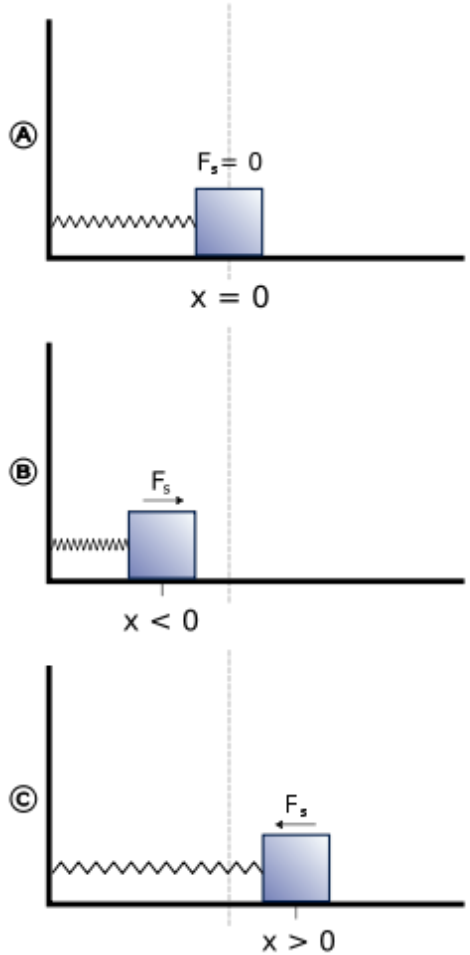
Displaying results

- Real-time or not?
- Timescale
- Response and interactivity
- Refresh rates

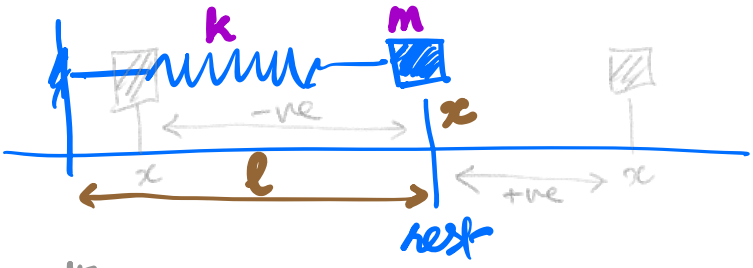
Mass spring system

Hook's law (1676) states, "the extension is proportional to the force."

Mathematically, $F = -kx$, where k is the spring constant and x is the displacement of the spring from rest position under the application of force F.



Simulate a mass-spring system.



Task: How the mass is going to move under the influence of spring.

k: spring constant

m: mass

x: current position of the mass

l: rest length of the spring.

x - l: deformation

$F = -kx$ ↑ deformation

$$F_{spring} = -K(x-l)$$

Force on mass due to spring.

$$F = ma$$

$$-k(x-l) = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}(x-l)$$

Reduce:

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -\frac{k}{m}(x - \ell)$$

Update rules:

$$\Delta x = v\,\Delta t$$

$$\Delta v = -\frac{k}{m}(x - \ell)\,\Delta t$$

State: x, v

Properties: k, m, ℓ

Mass spring system

Step 1: construct a model that will describe the motion of the mass over time

Hook's law: $F = -kx$

Newton's Second Law of Motion: $F = ma$

Combining the two we get

$$ma = -kx$$

$$\implies m\frac{dx^2}{dt^2} = -kx$$

Mass spring system

Step2: find a way to solve the model numerically

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Convert the second order $m\frac{dx^2}{dt^2} = -kx$ to a system of first order equations

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Convert the second order $m\frac{dx^2}{dt^2} = -kx$ to a system of first order equations

$$\frac{dx}{dt} = v$$
$$m\frac{dv}{dt} = -kx$$

And make the update rules

$$x(t + \Delta t) = x(t) + v(t)\Delta t$$
$$v(t + \Delta t) = v(t) - \frac{k}{m}x(t)\Delta t$$

new position & velocity

current position and velocity

Mass spring system

- Set values for mass m and spring constant k
- Set the initial conditions
  - Values for x and v at start time t_0
- Run the simulation loop
  1. Update t to $t + \Delta t$
  2. Update values for x and v using the update ruls
  3. Display results or save them to file for plotting
  4. Repeat steps 1 to 4

We just simulated a *Simple Harmonic Oscillator*

Code: `1d-mass-spring`

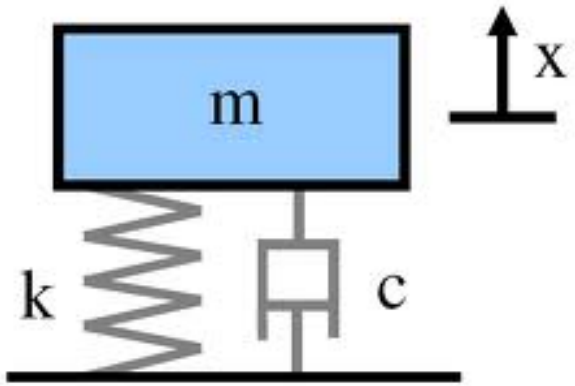
Mass spring system

```
# Mass-Spring system
class Mass:
    def __init__(self):
        self.x = 5
        self.vx = 0
        self.k = 1
        self.dt = 0.1
        self.t = 0
        self.m = 1.0

    def update(self):
        self.x += (self.vx * self.dt)
        self.vx += (- self.k * self.x * self.dt / self.m)
        self.t += self.dt
```

Mass Spring Damper

The mass experiences a damping force that is proportional to its current velocity



MT

Mathematically

$$F = -kx - cv$$

where c is the damping constant

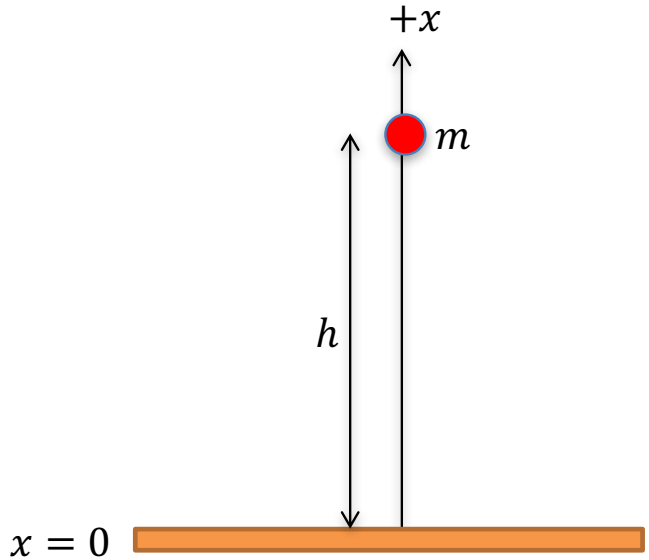
Code: modify `1d-mass-spring` to add damping effect

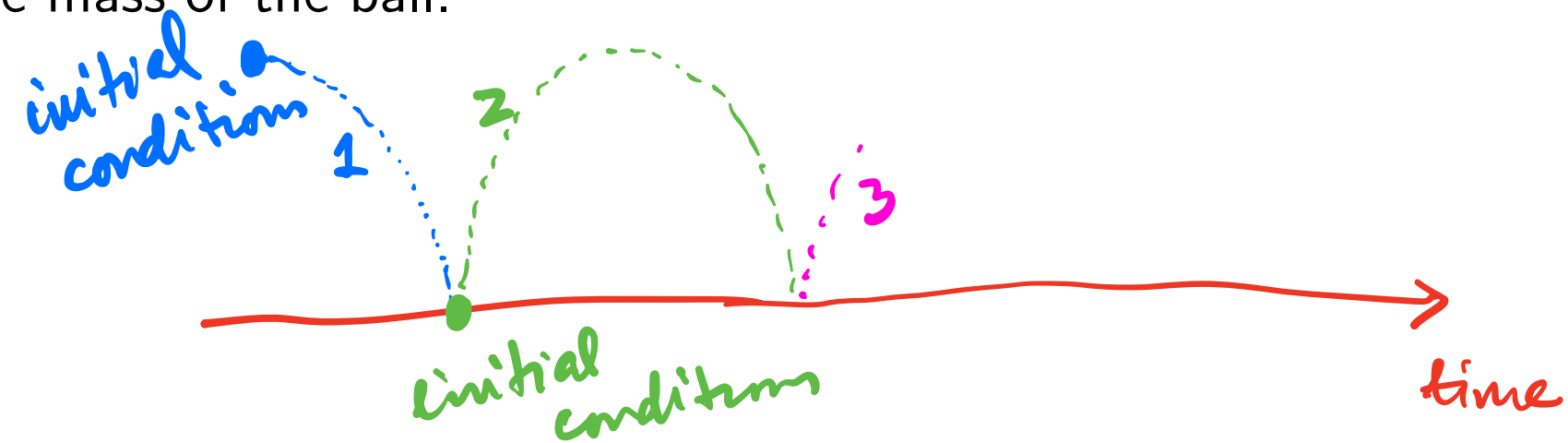
Bouncing ball

Assumption 1: We simplify the problem by treating the ball as a particle
From Newton's Second Law of Motion

$$F = m\frac{dx^2}{dt^2}$$

where x is the height of the ball from the ground and m is the mass of the ball.





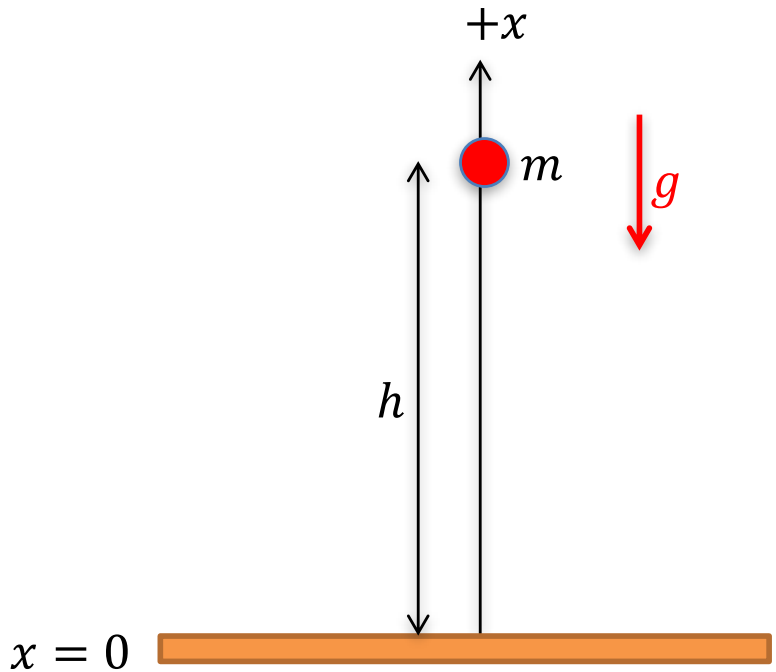
Bouncing ball

Assumption 2 Gravity is the only force acting upon this ball then

$$F = -mg$$

Putting it together we get

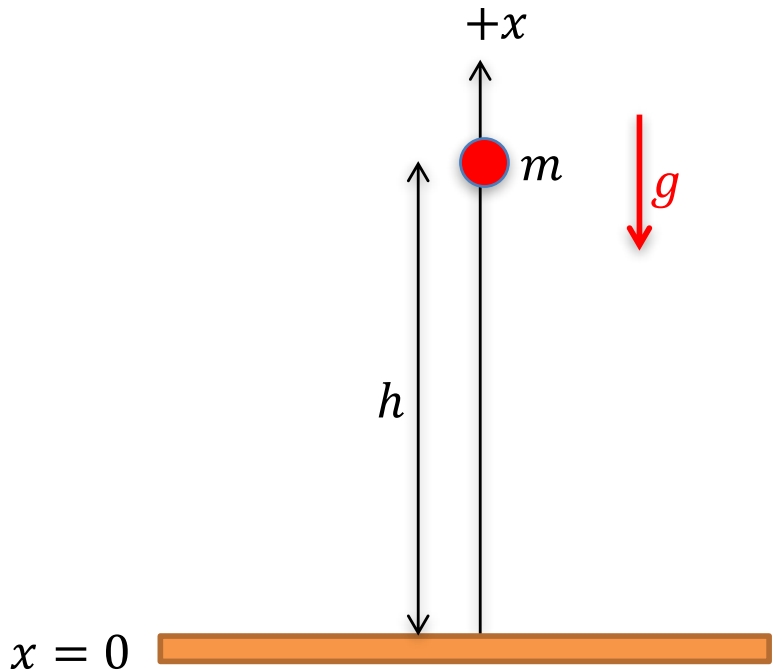
$$\frac{dx^2}{dt^2} = -g$$



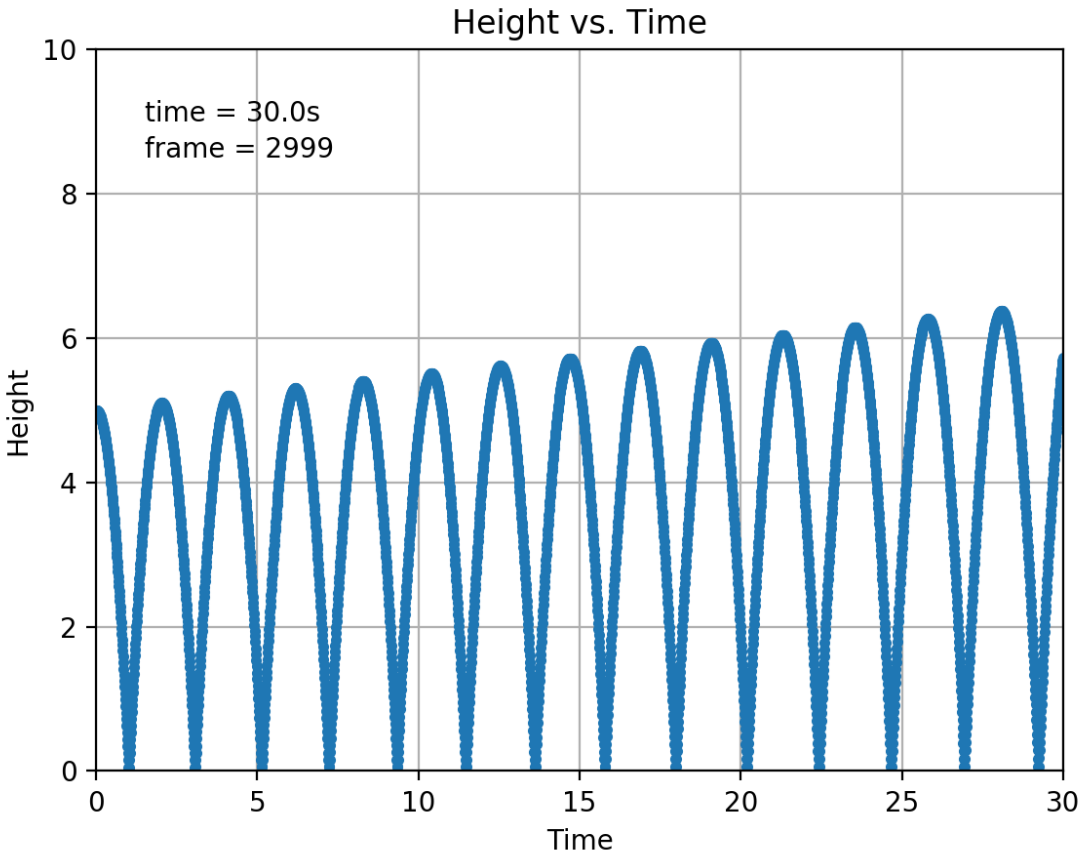
Bouncing ball

Data collection

- We need to know the value of g. For our purposes, we use $g = 9.8m/s^2$
- By using different g we can simulate bouncing ball on different planets



Bouncing ball



Did you notice something peculiar with this plot?

Code: `ball-floor` turn off RK4

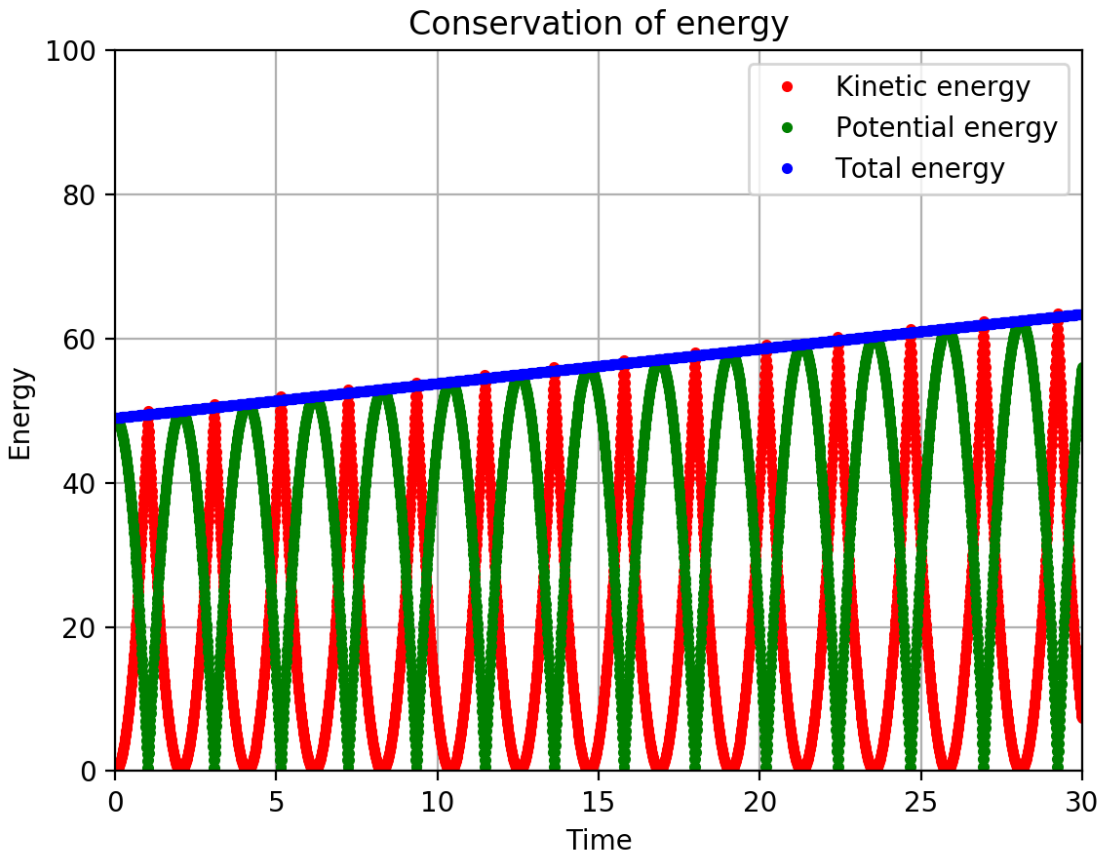
Bouncing ball

- The ball goes higher with each bounce, which is unexpected.
- The error doesn't go away even if we make timestep really small. It does, however, minimizes the effect.
- It seems we are imparting energy to the ball with each bounce. This breaks the *the law of conservation of energy*, which states that "the total energy of an isolated system remains constant."

Bouncing ball total energy

Total energy of the ball is the sum of its *kinetic* and *potential* energies.

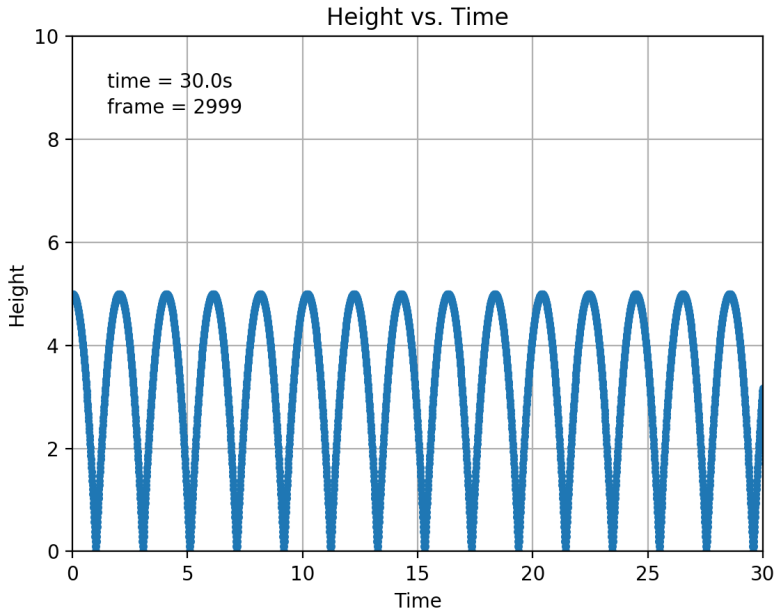
Kinetic energy $= \frac{1}{2}mv^2$
Potential energy $= mgy$

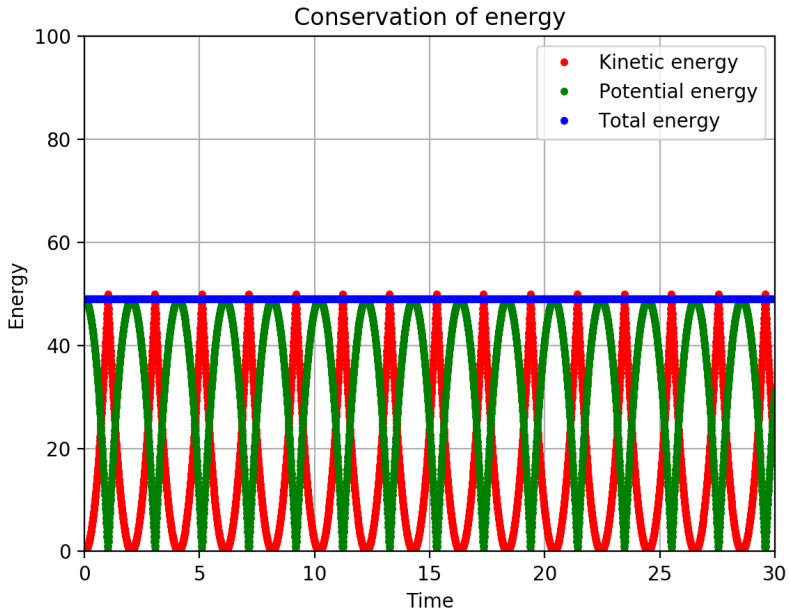


This behavior is due to incorrect assumptions of Euler method.

Bouncing Ball

Total energy is conserved when using Runga-Kutta or RK4 solver.





Code: `ball-floor` turn on RK4

Euler method

- A numerical solver for first order ODEs
- First order numerical procedure for solving ODEs (initial value problems)
- It is an *explicit method*
  - Calculates the state of the system at a later time given its current state by using the *update equations*
  - $y(t + \Delta t) = F(y(t))$

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Aside: implicit methods

- Calculates the state of the system at a later time given by solving an equation that includes both the future state and the current state
- $G(y(t + \Delta t), y(t)) = 0$

Euler method

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  - Adversally effects accuracy
  - Exhibits error growth over time
    - Error is proportional to Δt

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- Particularly unsuited for stiff equations
  - Equations containing terms that lead to rapid changes
  - E.g., a mass spring system with large spring constant
- Use *extremely* small time steps
  - Infeasible in practice

Runga-Kutta method

- An other numerical solver for first order ODEs
- An alternate to Euler method
- A family of explicit and implicit methods
- Often RK4 is used
  - Error is proportional to Δt^4
  - Makes a huge difference for small values of Δt

Takeaway: whenever possible use RK4 method

Numerical solvers in Python

```
from scipy.integrate import ode
def f(self, t, y, arg1):
    """Solves y' = f(t, y)
       Arguments:
       - y is the state of the system.  In our case
         y[0] is the position and y[1] is the velocity.
       - arg1 is 9.8, as set by set_f_params() method.

       Returns vector dy/dt.  In our case, dx/dt = v and
       dv/dt = -g.
       """
    return [y[1], -arg1]

r = ode(f).set_integrator('dop853')
r.set_initial_value([y0, vy0], t0)
r.set_f_params(9.8)
r.integrate(dt)
print r.t, r.y
```

Bouncing ball: takeaways

- Exploit your knowledge of physics to determine if simulation is behaving as expected
- Use several strategies
- Compare outputs of several strategies
  - If outputs differ, you must have a way to explain the differences
  - If outputs are the same, the simulation *may* be correct

Discussion

Q. Why does Euler method performing so poorly for our bouncing ball example?

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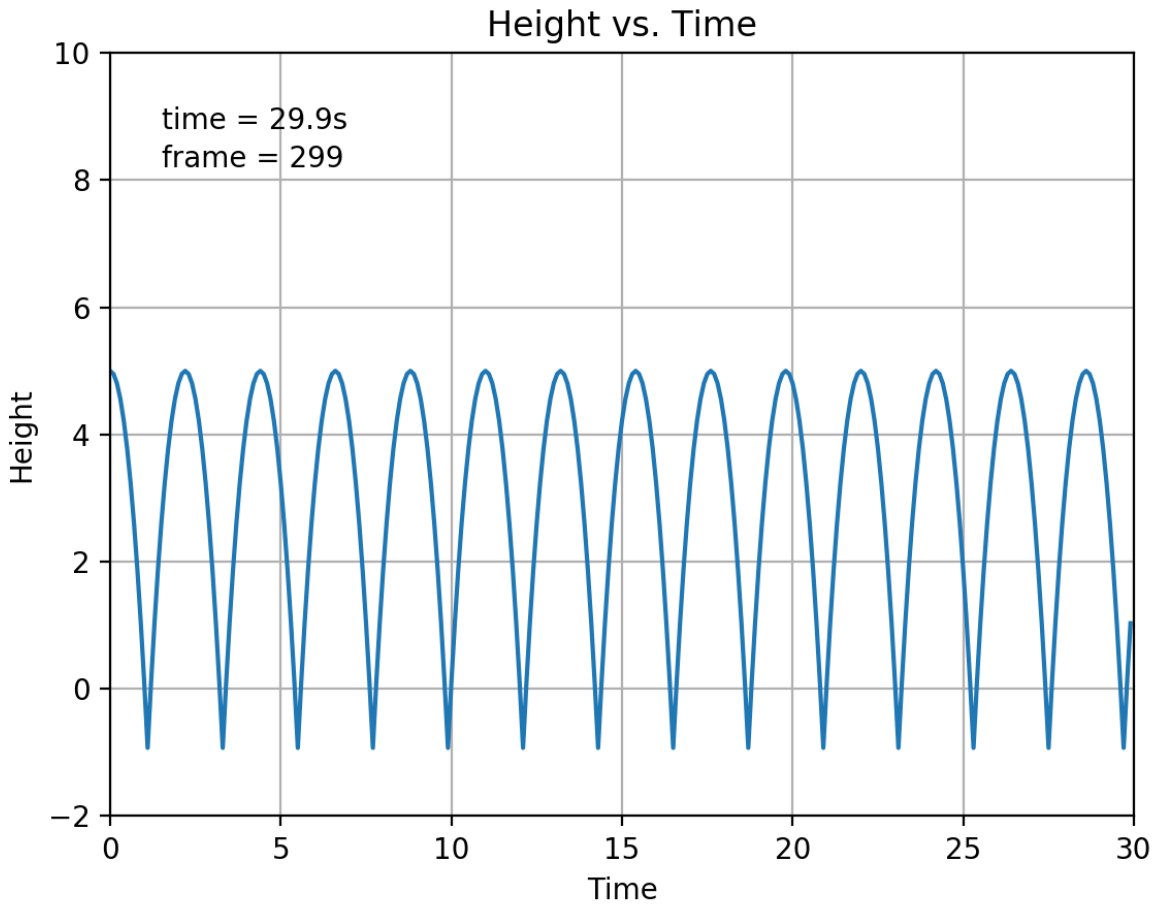
Q. Why does Euler method performing so poorly for our bouncing ball example?

A. Euler method assumes that the acceleration remains constant between two time steps. Notice that this assumption is generally false, but especially so when the ball "hits" the ground at $x = 0$. The velocity is flipped, changing the sign of the derivative and causing a discontinuity.

RK4 method is much better at handling discontinuities (as long as there aren't too many of these).

This is why RK4 is able to get good results even for large time steps.

Bouncing ball



For this simulation, the floor sits at height 0. The ball pierces through the floor, which is incorrect.

Code: `ball-floor` increase timestep to see the ball penetrating the floor

Bouncing ball

Need a better way to detect collisions with the floor

Bouncing ball

Need a better way to detect collisions with the floor

Scheme 1

- Use smaller time steps
- The ball will travel less distance between two time steps, and there is a greater chance of catching the collision instant
- In any case, the ball will penetrate less into the floor

Bouncing ball

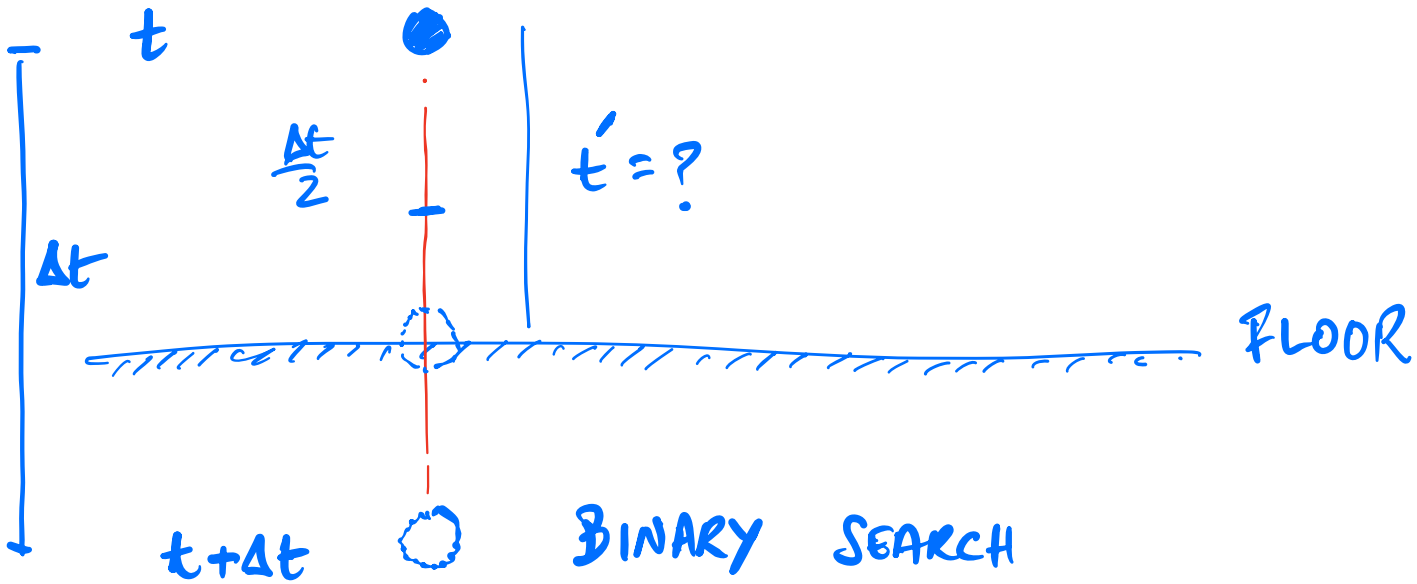
Need a better way to detect collisions with the floor

Scheme 1

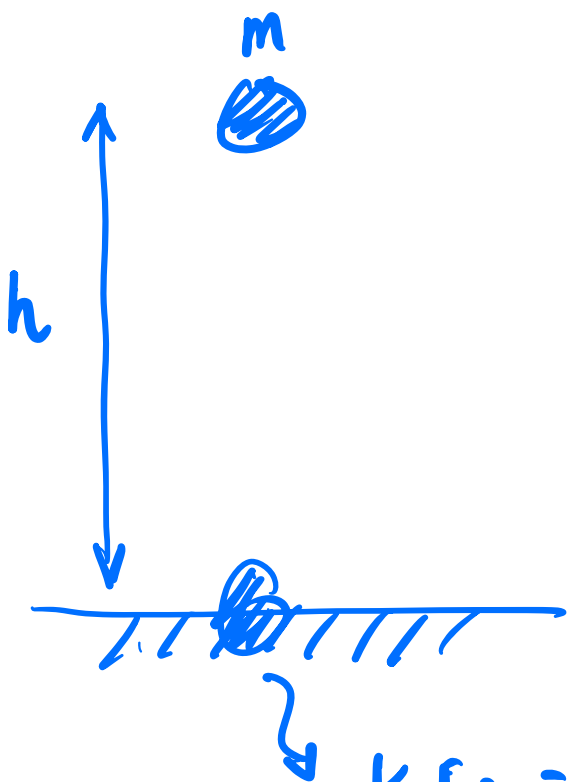
- Use smaller time steps
- The ball will travel less distance between two time steps, and there is a greater chance of catching the collision instant
- In any case, the ball will penetrate less into the floor

Scheme 2

- Try to find the exact time of collision using $x = vt$ relationship
- Adjust time step accordingly



BINARY SEARCH



$K.E. = \frac{1}{2}mv^2$

$P.E. = mgh$

$K.E. = P.E.$

$\frac{1}{2}mv^2 = mgh$

$\Rightarrow v^2 = 2gh$

$\Rightarrow v = \sqrt{2gh}$