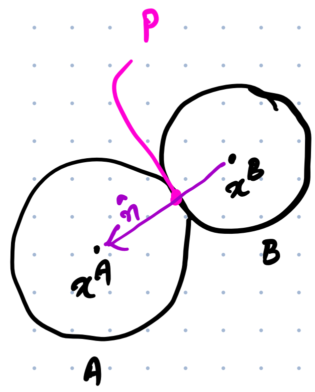
March 4, 2026

* Rigid Body Collisions



Consider two balls A and B

Position of ball A : x^A
" B : x^B

Radius of ball A : r^A
" B : r^B

Velocity of ball A : v^A
" B : v^B

Mass of ball A : m^A
" B : m^B

There are no rotational effects.

Velocity of collision point P for ball A : $v^{AP} = v^A$
B : $v^{BP} = v^B$

* Collision detection

(i) compute normal $\hat{n} = (x^A - x^B)/|x^A - x^B|$

(ii) compute relative velocity $v^{AB} = v^{AP} - v^{BP} = v^A - v^B$

$v^{AB} \cdot \hat{n} = \phi \longrightarrow$ resting contact
$< \phi \longrightarrow$ collision is imminent
$> \phi \longrightarrow$ A and B are moving away from each other

* Collision Response

① Use Newton's Law of Restitution for Instantaneous Collision with No Friction.

impulse

impulse: an infinite force applied for a very short duration.

this force is equal to the change in momentum.

$$J = \Delta P$$

impulse

change in momentum.

? A body of mass m has velocity v_1 before collision and v_2 after collision. What is its change in momentum?

$$J = mv_1 - mv_2$$

momentum before collision

momentum after collision.

$$\Rightarrow v_2 = v_1 - \frac{J}{m}$$

Assumptions:

- no gravity
- no friction
- conservation of momentum. J for first body is equal to $-J$ of the second body
- can only deal with two body collisions at any given time.

② Empirical model of frictionless collisions

$$v_2^{AB} \cdot \hat{n} = -e \, v_1^{AB} \cdot \hat{n}$$

Relative velocity between A and B after collision.

Relative velocity between A and B before collisions.

Relative velocity between the two bodies A and B after collision (along the collision direction $\hat{n}$) is a function of the relative velocity before collision.

e is called the co-efficient of restitution.

- $e = 1$ elastic collision, no loss of K.E.
- $e = 0$ perfectly inelastic collision, total loss of K.E.
- $0 < e < 1$ some loss of K.E.

* Given ① and ② above, lets solve for the velocities of balls A and B after collision.

v_1^{AP} : velocity of P in A before collision.

v_1^{BP} : " B "

v_2^{AP} : " A after collision

v_2^{BP} : " B "

From ①

impulse

Ⓐ ~ $v_2^{AP} = v_1^{AP} + \frac{j\hat{n}}{m^A}$

collision direction

Ⓑ ~ $v_2^{BP} = v_1^{BP} - \frac{j\hat{n}}{m^B}$

j : impulse of A

$-j$: " B

Subtract Ⓑ from Ⓐ

$$(v_2^{AP} - v_2^{BP}) = (v_1^{AP} - v_1^{BP}) + \left[\frac{1}{m_A} + \frac{1}{m_B}\right] j\hat{n}$$

$$v_2^{AB} = v_1^{AB} + \left[\frac{1}{m_A} + \frac{1}{m_B}\right] j\hat{n} \quad \text{—— ⓒ}$$

relative velocity before collision

relative velocity after collision.

From ②

$$v_2^{AB}\cdot n = -e\, v_1^{AB}\cdot n \quad \text{(D)}$$

Using (C) and (D)

$$-e\, v_1^{AB}\cdot n = v_1^{AB}\cdot n + \left[\frac{1}{m^A}+\frac{1}{m^B}\right] \underbrace{j}_{?}\; \underset{1}{\underline{\underline{n\cdot n}}}$$

$$\Rightarrow \quad j = \frac{-(1+e)\, v_1^{AB}\cdot n}{\left[\frac{1}{m^A}+\frac{1}{m^B}\right]}$$

Strategy: Given $\underbrace{v_1^A, v_1^B, m^A, m^B, e, x^A, x^B}$

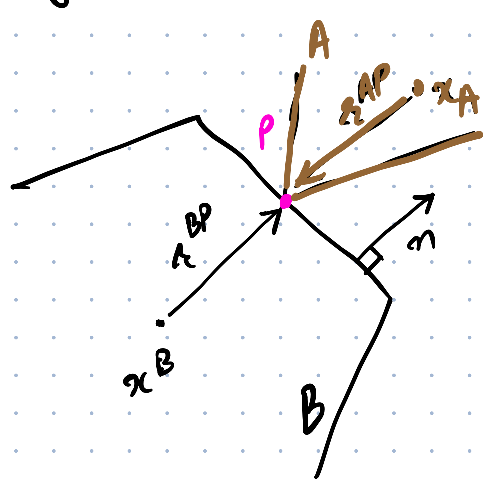
check for collision.

compute j for A. $\longrightarrow$ $-j$ for B.

$$v_2^A = v_1^A - \frac{j\hat{n}}{m^A}$$

$$v_2^B = v_2^A + \frac{j\hat{n}}{m^B}$$

* Rigid Body Collisions with Rotational Effects.



Collision will be felt along $\hat{n}$.

collision point P in A: r^{AP}

" P " B: r^{BP}

Before collision:

velocity of P in A: $v_1^A + \omega_1^A \times r^{AP}$

B: $v_1^B + \omega_1^B \times r^{BP}$

linear

angular

we want to compute v_2^A, v_2^B, ω_2^A, and ω_2^B.

Mass of body A: m^A Inertia tensor: I^A (world)

B: m^B " I^B "

We know already

$$v_2^A = v_1^A + \frac{j\hat{n}}{m^A}$$

$$\omega_2^A = \omega_1^A + (I^A)^{-1} r^{AP} \times j\hat{n}$$

Similarly for body B

$$v_2^B = v_1^B - \frac{j\hat{n}}{m^B}$$

$$\omega_2^B = \omega_1^B - (I^B)^{-1} r^{BP} \times j\hat{n}$$

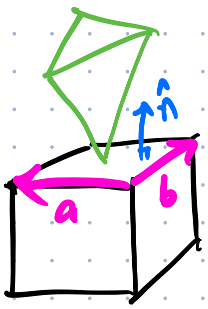
Use these to compute j

$$j = \frac{-(1+e)\, v^{AB} \cdot \hat{n}}{\left[\frac{1}{m^A} + \frac{1}{m^B}\right] + n \cdot (I^A)^{-1}(r^{AP} \times n) \times r^{AP} + n \cdot (I^B)^{-1}(r^{BP} \times n) \times r^{BP}}$$

Strategy:
(i) compute n
(ii) compute j
(iii) v_2^A, ω_2^A, v_2^B, ω_2^B

(iv) Update the state of two bodies

(v) continue simulation.



$$\vec{b} \times \vec{a} = \hat{n}$$