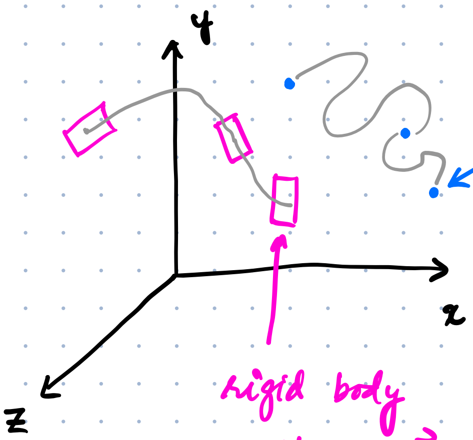
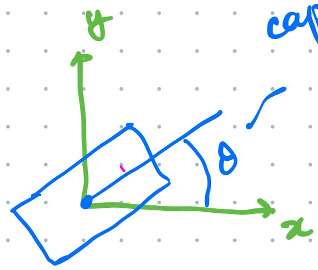
February 4, 2026

* Rigid Bodies



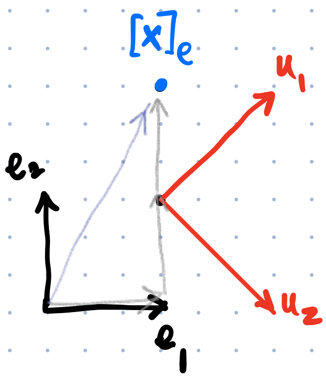
particle
position $\vec{p} \in \mathbb{R}^3$
velocity $\vec{v} \in \mathbb{R}^3$

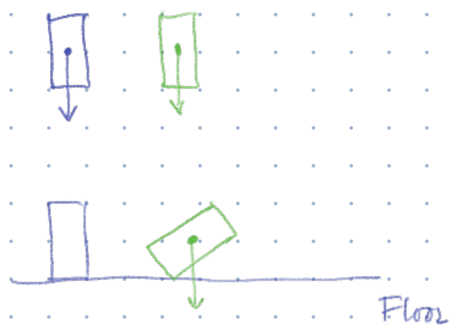
rigid body
position $\vec{p}$
velocity $\vec{v}$
orientation θ
angular velocity ω (how fast is it spinning)



captures orientation in 2D.

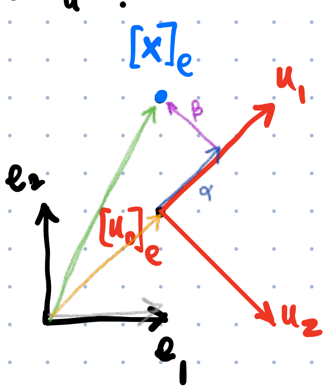
Coordinate Systems





$[x]_e = 1\vec{e}_1 + 2\vec{e}_2 = (1,2)$

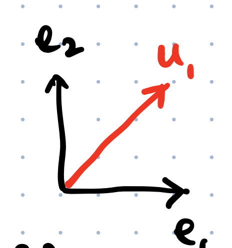
Q. What is $[x]_u$?

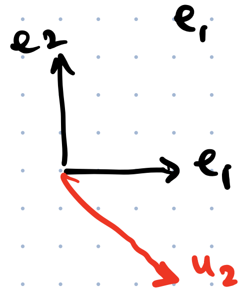


(i) $[u_0]_e = (1,1)$

(ii) $[u_1]_e = (1,1)$

(iii) $[u_2]_e = (1,-1)$





$$[u_0]_e + \alpha[u_1]_e + \beta[u_2]_e = [x]_e$$

$$\alpha[u_1]_e + \beta[u_2]_e = [x]_e - [u_0]_e$$

$$\underbrace{\begin{bmatrix}[u_1]_e & [u_2]_e\end{bmatrix}}_{\in \mathbb{R}^{2\times 2}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = [x]_e - [u_0]_e$$

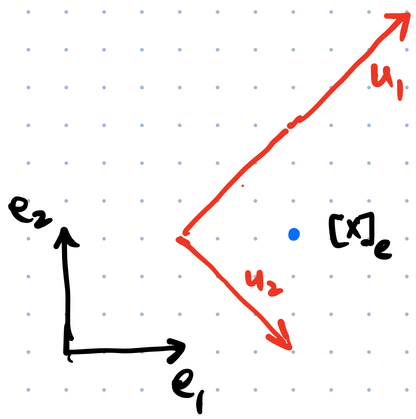
(α, β) are the coordinates of $[x]_e$ in $u_1 u_2$ coordinate system.

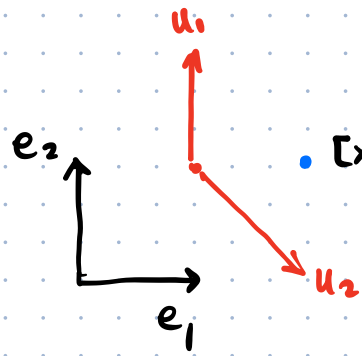
$$\underbrace{\begin{bmatrix}\alpha\\ \beta\end{bmatrix}}_{[x]_u} = \begin{bmatrix}[u_1]_e & [u_2]_e\end{bmatrix}^{-1}\left([x]_e - [u_0]_e\right)$$

$$\alpha(a, b) + \beta(c, d) = \begin{pmatrix}\alpha a + \beta c\\ \alpha b + \beta d\end{pmatrix}$$

$$\begin{bmatrix}a & c\\ b & d\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{pmatrix}a\alpha + c\beta\\ b\alpha + d\beta\end{pmatrix}$$

Exercise :

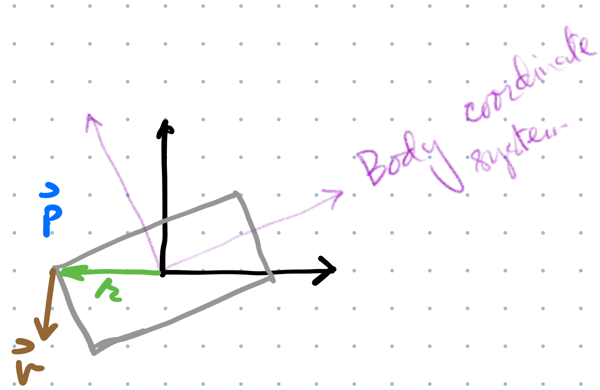




TAKE HOME EXERCISE.

* Rigid Bodies in 2D

- angular velocity: ω
  units are radians per second



- Radian is the angle subtended by an arc whose length is equal to its radius: $\theta = \frac{l}{r}$.

- linear velocity at a point on the body.

$$v = r\omega$$

* Rigid Bodies in 3D

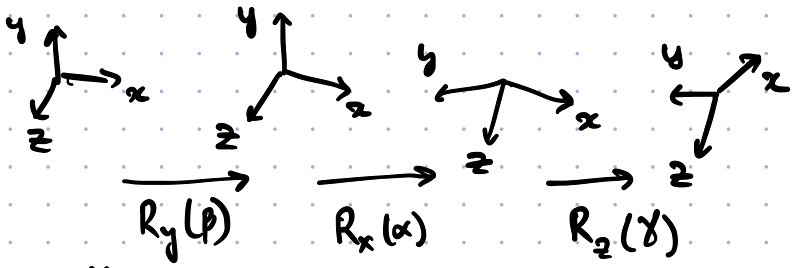
(i) 3D rotations cannot be described using an angle.

(ii) There are many ways to represent three-dimensional rotations.

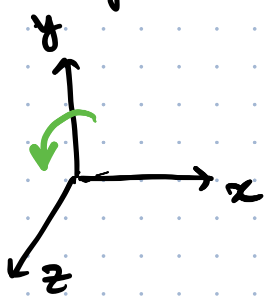
(a)  axis angle

(b) Quaternions $\in \mathbb{R}^4$

(c) 3 rotations around axis.



$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$



$$R_y(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

$$\vdots$$

* Represent rotations in 3D as a 3x3 matrix R.

(i) R is an orthogonal matrix

(ii) $RR^T = I \longrightarrow R^T = R^{-1}$

$$\begin{bmatrix} \rightarrow \\ \rightarrow \\ \rightarrow \end{bmatrix}\begin{bmatrix} \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* Forces on the rigid body

- Net force acting on an object is the rate of change of its linear momentum.

$$\frac{dP}{dt} = F \leftarrow \text{force}$$

- Linear momentum: $\underline{P = mv}$

$$\frac{dmv}{dt} = F$$

$$\frac{m\,dv}{dt} = F$$

$$\boxed{ma = F}$$

- Torque acting on a rigid body: net torque acting on an object about point o is the rate of change of its angular momentum

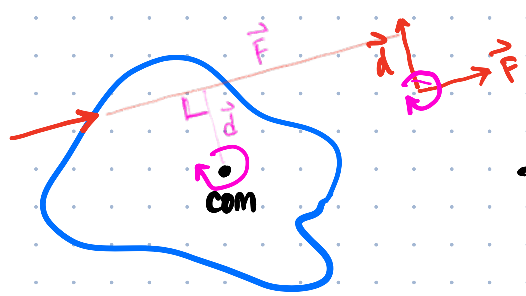
$$\frac{dL}{dt} = N \leftarrow \text{torque}$$

- angular momentum: $L = I\omega$

inertia tensor

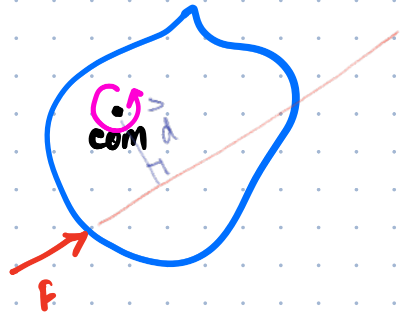
angular velocity

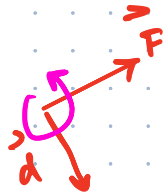
* Torque

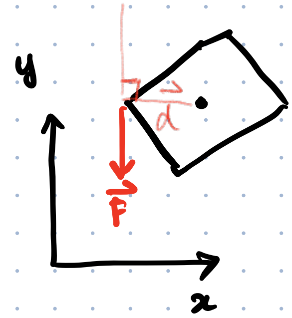


com: centre of mass

$$T = \underbrace{d \times F}_{\text{cross-product}}$$







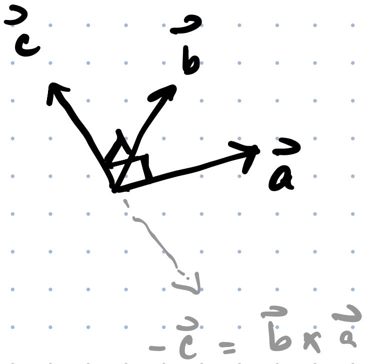
Cross-product:

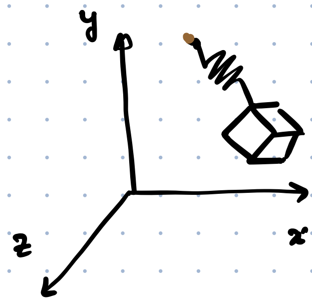
$$\vec a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad \vec b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\vec a \times \vec b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$\vec c \perp \vec a$ and $\vec c \perp \vec b$

$\vec c = \vec a \times \vec b$





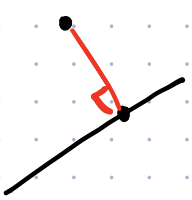
* Computing Torque

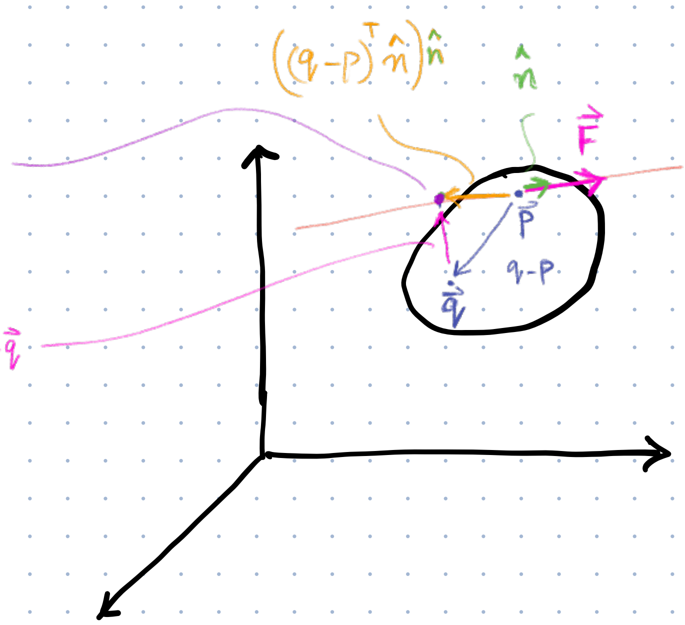
$$\hat n = \frac{\vec F}{\|F\|}$$

$p - ((q-p)^T \hat n)\hat n$

$\vec d = \left(p - ((q-p)^T \hat n)\hat n\right) - \vec q$

$T = \vec d \times \vec F$





* Dot Product

$\vec{v} = (a, b)$

$\vec{w} = (c, d)$

$\vec{v} \cdot \vec{w} = ac + bd$

$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad$ $\vec{v}^T = [a \;\; b]$

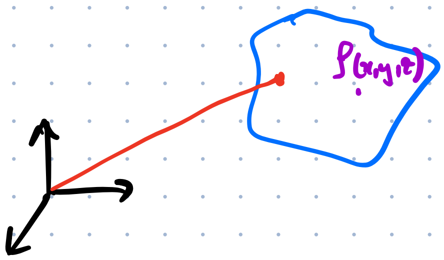
$\vec{w} = \begin{bmatrix} c \\ d \end{bmatrix}$ $\quad$ $\vec{w}^T = [c \;\; d]$

$$\vec{v}^T \vec{w} = \underset{1\times 2}{[a \;\; b]} \underset{2\times 1}{\begin{bmatrix} c \\ d \end{bmatrix}} = [ac + bd]$$

dot-product

inner-product

* Centre of Mass (COM)



(i) COM is the mean location of all the mass in the body.

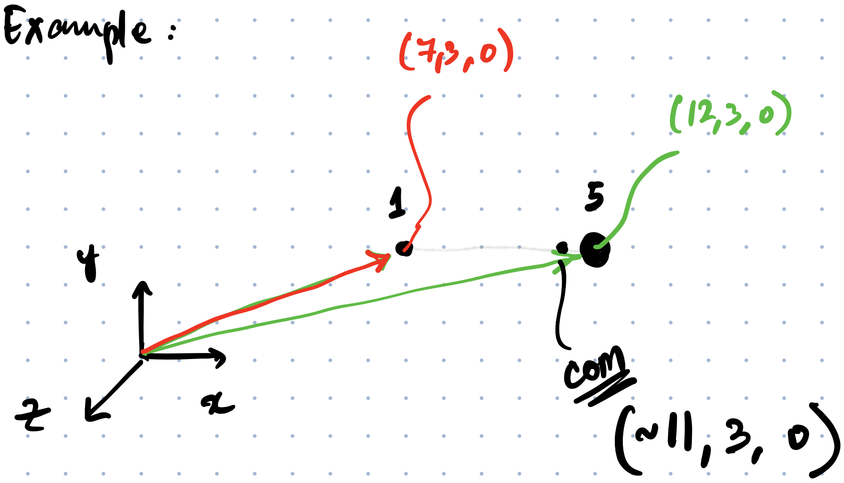
(ii) $r_x = \frac{1}{M} \int \rho(x,y,z)\, x \, dV$

$r_y = \frac{1}{M} \int \rho(x,y,z)\, y \, dV$

$r_z = \frac{1}{M} \int \rho(x,y,z)\, z \, dV$

(iii) M = total mass

Example :



One square is 1 unit length.

$$q_x = \frac{(7)(1) + (12)(5)}{1+5} = \frac{67}{6} \approx 11.\ldots$$

$$q_y = \frac{(3)(1) + (3)(5)}{1+5} = \frac{3+15}{6} = \frac{18}{6} = 3$$

$$q_z = \frac{(0)(1) + (0)(5)}{1+5} = 0$$

Example: