Polynomial Approximation

Computational Photography (CSCI 3240U)

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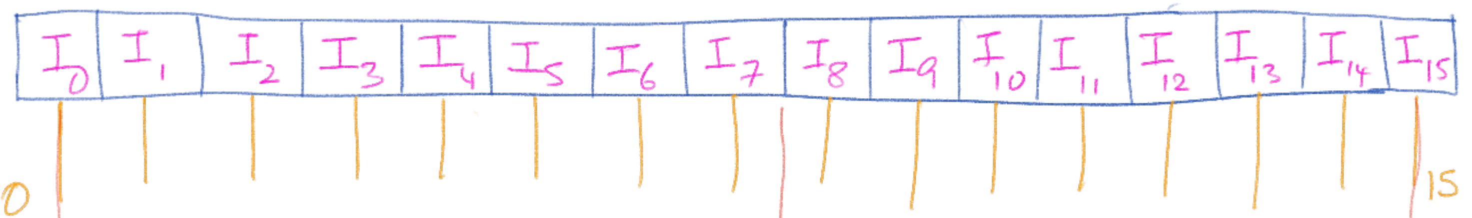
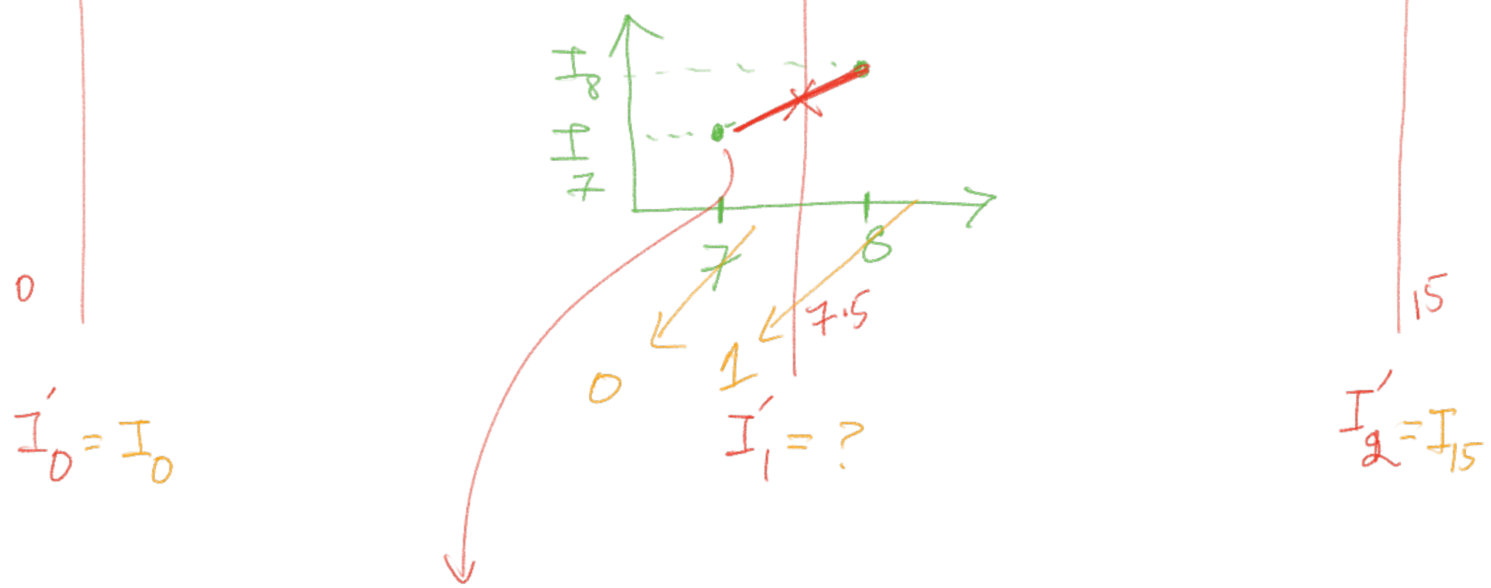


\# pixels = 3

locations for the new pixels : $\frac{(16-1)}{(3-1)} = \frac{15}{2} = 7.5$



$(0, I_7)$ and $(1, I_8)$

$$\Rightarrow \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \frac{y - I_7}{I_8 - I_7} = \frac{x - 0}{1 - 0}$$

$$\Rightarrow y - I_7 = x(I_8 - I_7)$$

$$\Rightarrow \quad y = x\left(I_8 - I_7\right) + I_7$$

$$\Rightarrow \quad y = \frac{1}{2}\left(I_8 - I_7\right) + I_7$$

$$\Rightarrow \quad y = \frac{1}{2} I_8 - \frac{1}{2} I_7 + I_7$$

$$\Rightarrow \quad y = \frac{1}{2} I_8 + \frac{1}{2} I_7$$

$$\boxed{\Rightarrow y = \frac{I_7 + I_8}{2}}$$

Today’s lecture

- How to compute image derivatives by fitting polynomials to 1D image patches?
  - Taylor series expansion around a patch center
  - Least square fitting of a system of linear equations

Image as a surface in 3D

Consider a gray-scale image $I(x, y)$ then the height of the surface at (x, y) is $I(x, y)$. The surface passes through the 3D point $(x, y, I(x, y))$.

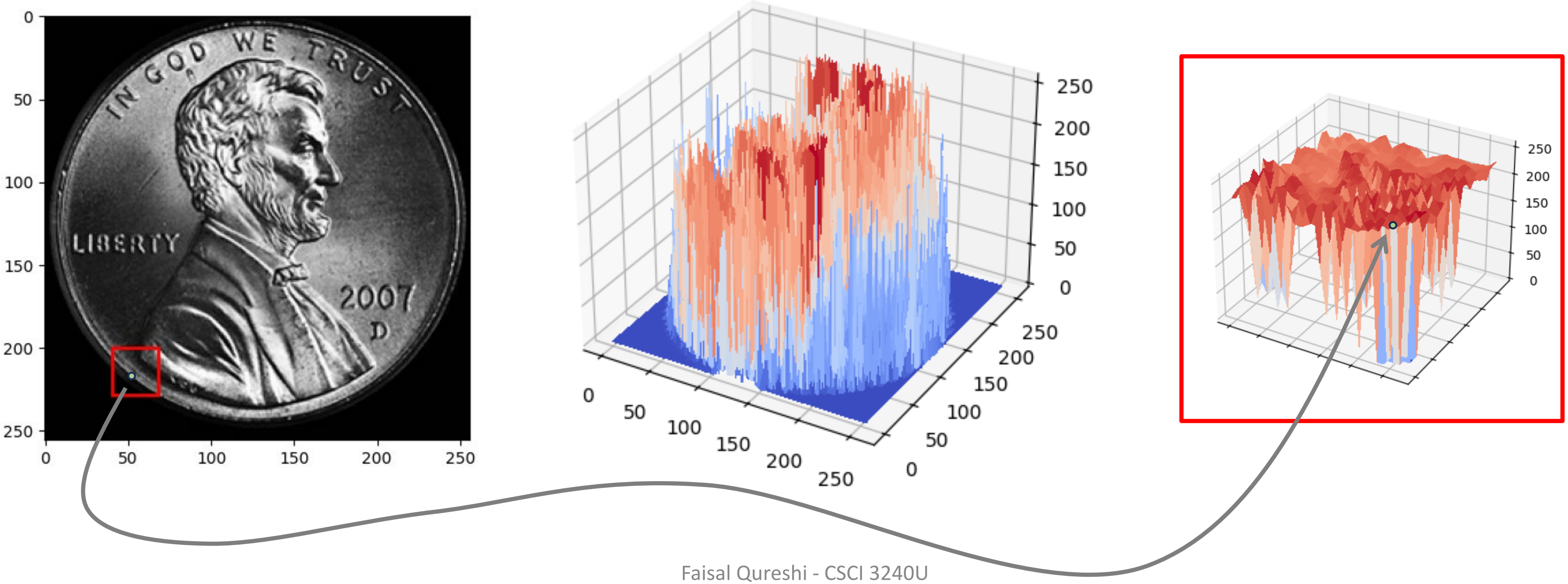
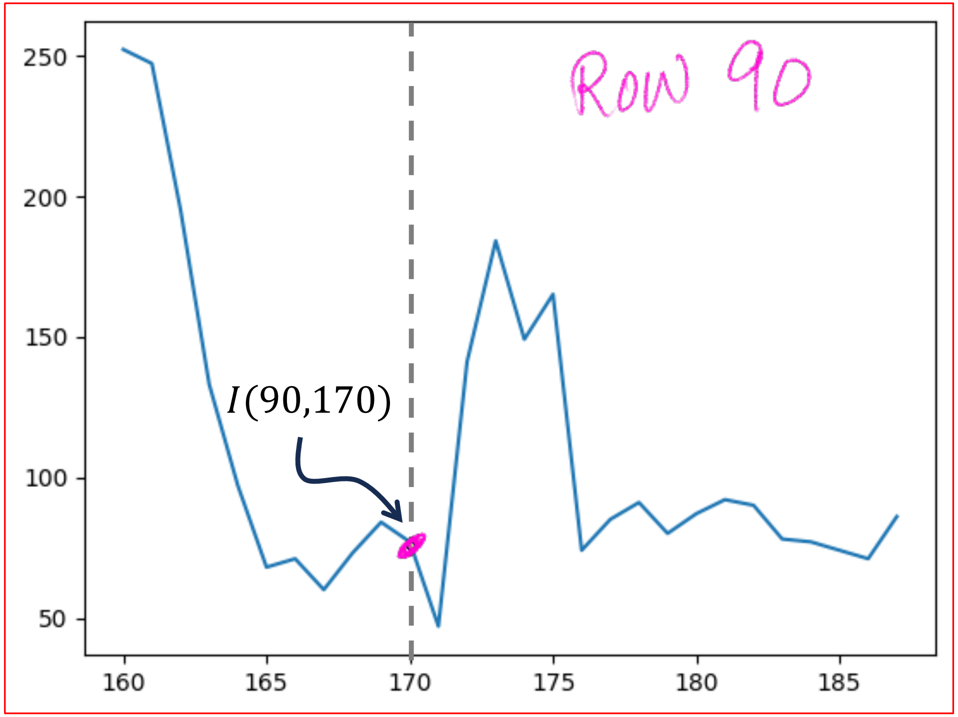


Image rows (or columns) as 2D graphs



Paths as curves in 2D

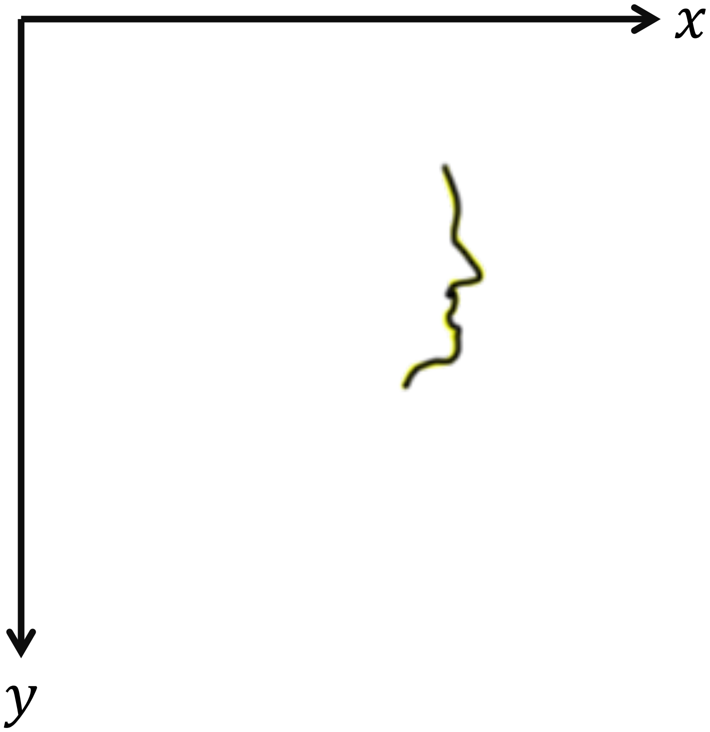
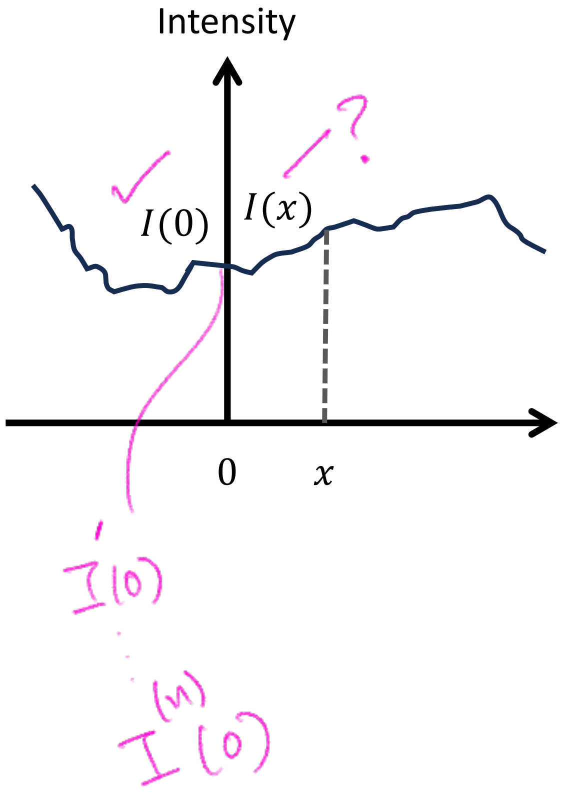


Image rows (or columns) as 2D graphs

Polynomial approximation



Taylor series expansion of $I(x)$ near the "patch" center 0

$$I(x) = I(0) + xI'(0) + \frac{x^2}{2!}I''(0) + \frac{x^3}{3!}I'''(0) + \cdots + \frac{x^n}{n!}I^{(n)} + R_{n+1}(x)$$

0th order

1st order

2nd order

Factorials

$n! = n(n-1)\cdots 1$

$0! = 1$

The residual $R_{n+1}(x)$ satisfies:

$$\lim_{x\to 0} R_{n+1}(x) = 0$$

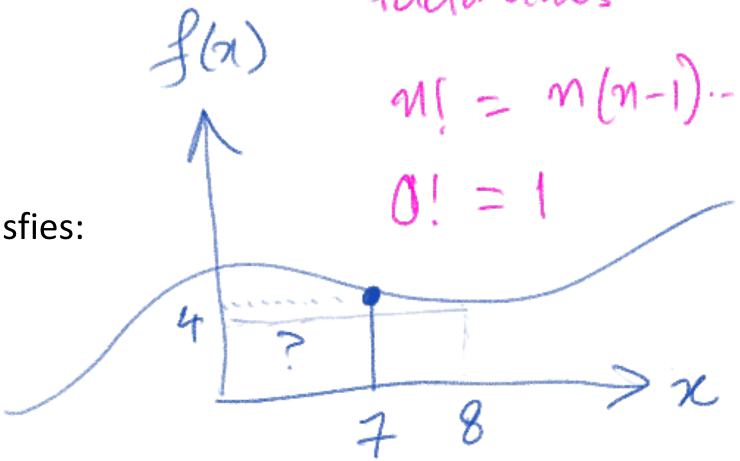
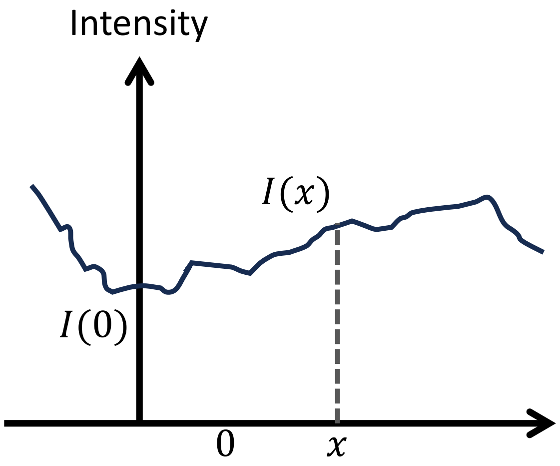


Image rows (or columns) as 2D graphs

Polynomial approximation

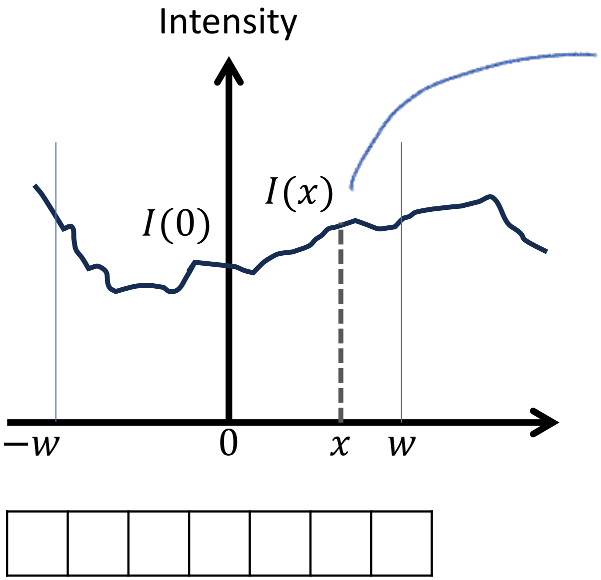


Taylor series expansion of $I(x)$ near the “patch” center 0

$$I(x) = \underbrace{I(0) + xI'(0) + \frac{x^2}{2!}I''(0) + \frac{x^3}{3!}I'''(0) + \cdots + \frac{x^n}{n!}I^{(n)}}_{\text{Nth order approximation}} + \underset{\text{Error}}{\underline{\underline{R_{n+1}(x)}}}$$

For a given x, approximation depends on $(n+1)$ constants corresponding to the intensity derivative at the patch origin.

Polynomial approximation



Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \underset{\text{approx}}{\underline{\approx}} I(0) + xI'(0) + \frac{1}{2}x^2I''(0) + \frac{1}{6}x^3I'''(0) + \cdots + \frac{1}{n!}x^nI^{(n)}(0)$$

Re-write in matrix form

$$I(x) \approx \underbrace{\begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \cdots & \frac{1}{n!}x^n \end{bmatrix}}_{\in \mathbb{R}^{1\times(n+1)}} \underbrace{\begin{bmatrix} I(0) \\ I'(0) \\ I''(0) \\ I'''(0) \\ \vdots \\ I^{(n)}(0) \end{bmatrix}}_{\in \mathbb{R}^{(n+1)\times 1}}$$

location x

All the information about the fn. at location 0.

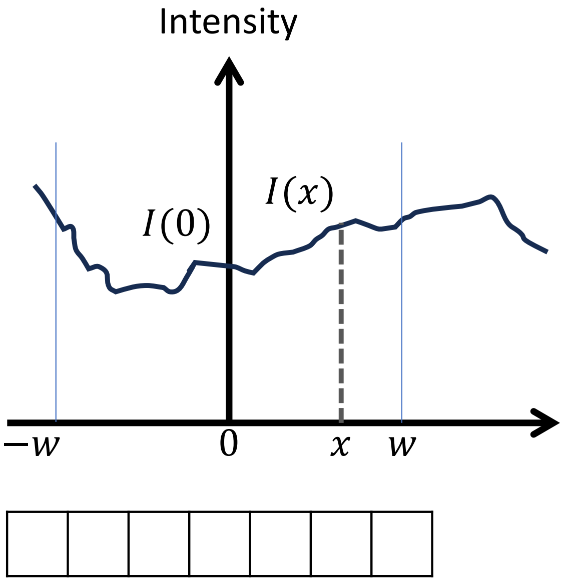
Polynomial approximation

MEMORIZE ↑

Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2I''(0) + \frac{1}{6}x^3I'''(0) + \cdots + \frac{1}{n!}x^nI^{(n)}(0)$$

— Ⓐ



Re-write in matrix form

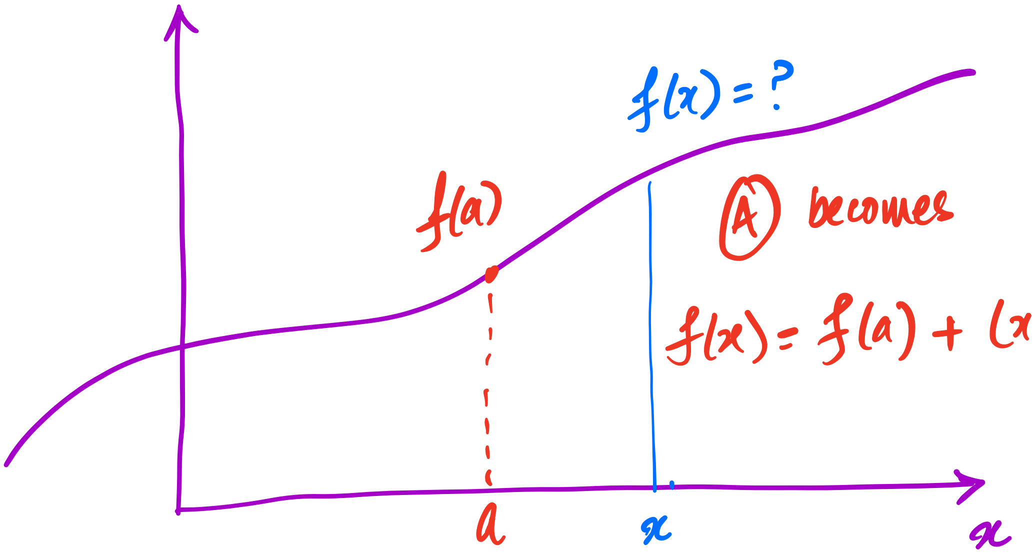
✓

$$I(x) \approx \begin{bmatrix}1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \dots & \frac{1}{n!}x^n\end{bmatrix} \left[\begin{array}{c} I(0)\\ I'(0)\\ I''(0)\\ I'''(0)\\ \vdots \\ I^{(n)}\end{array}\right] \begin{array}{c} d_0\\ d_1\\ d_2\\ d_3\\ \\ d_n\end{array}$$

↑

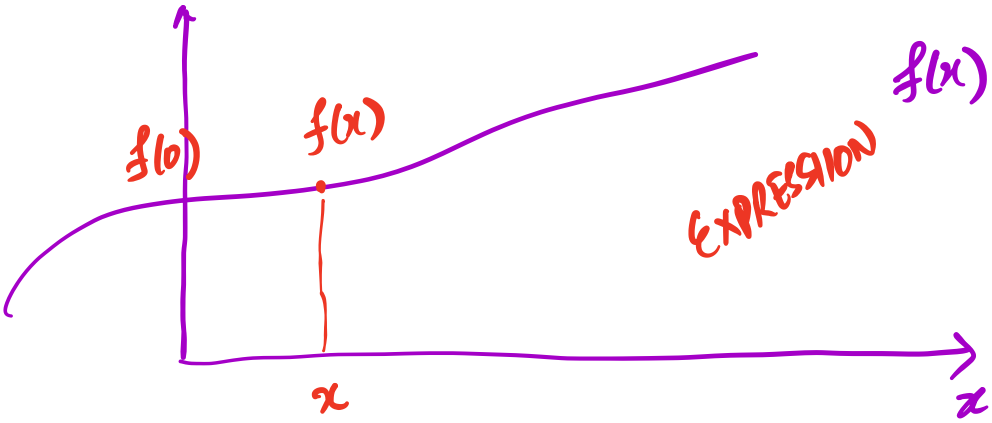
For notational simplicity, lets refer the vector of intensity and its derivatives as $\boldsymbol{d}$

Oct 27, 2023 Quiz



Ⓐ becomes

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2 f''(a)/2$$

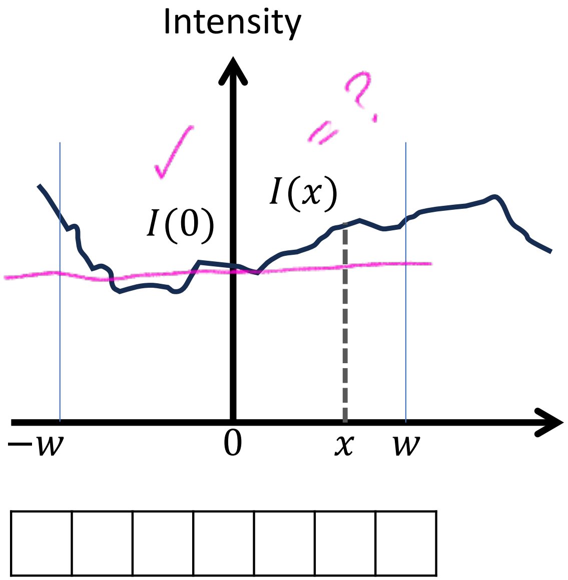


EXPRESSION

$$f(x) = f(0) + x f'(0) + x^2 f''(0)/2$$

$$= f(0) + (x-0)f'(0) + (x-0)^2 f''(0)/2$$

Polynomial approximation



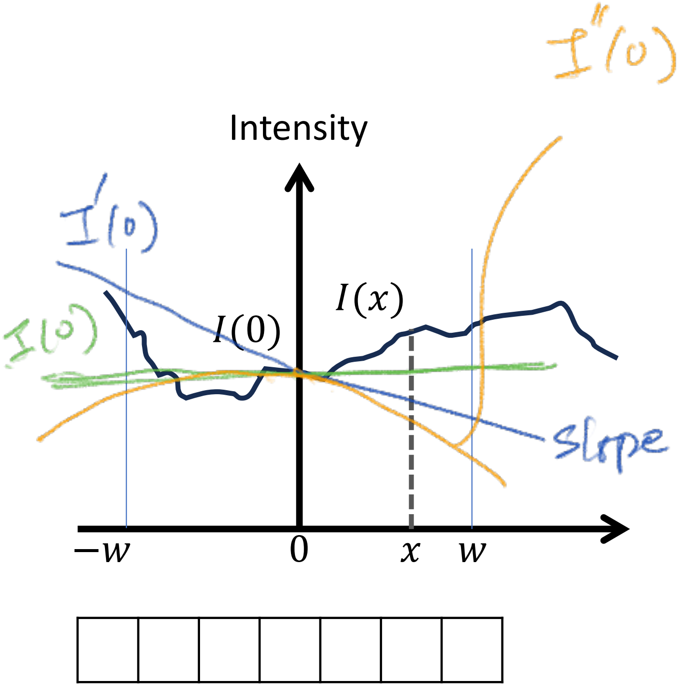
Taylor series expansion of $I(x)$ near the patch center 0

$$I(x) \approx I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Example

Show the 0th order approximation

Polynomial approximation



$= \frac{\Delta y}{\Delta x}$

Taylor series expansion of $I(x)$ near the patch center 0

$$\underline{I(x) \approx \underline{I(0)} + \underline{xI'(0)} + \frac{1}{2}x^2 \underline{I''(0)}} + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

Practice Question

Show the 1st and 2nd order approximations

$y = \underline{m}x + b$

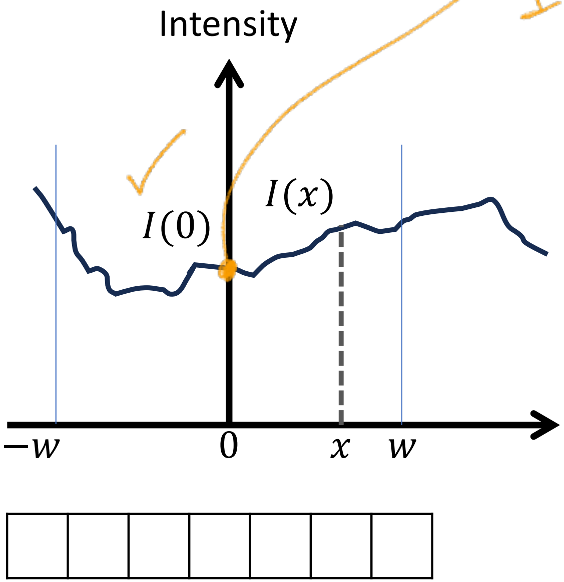
?

$\frac{\Delta y}{\Delta x}$

Compute derivatives at pixel 0 (i.e., the center of the pathc)

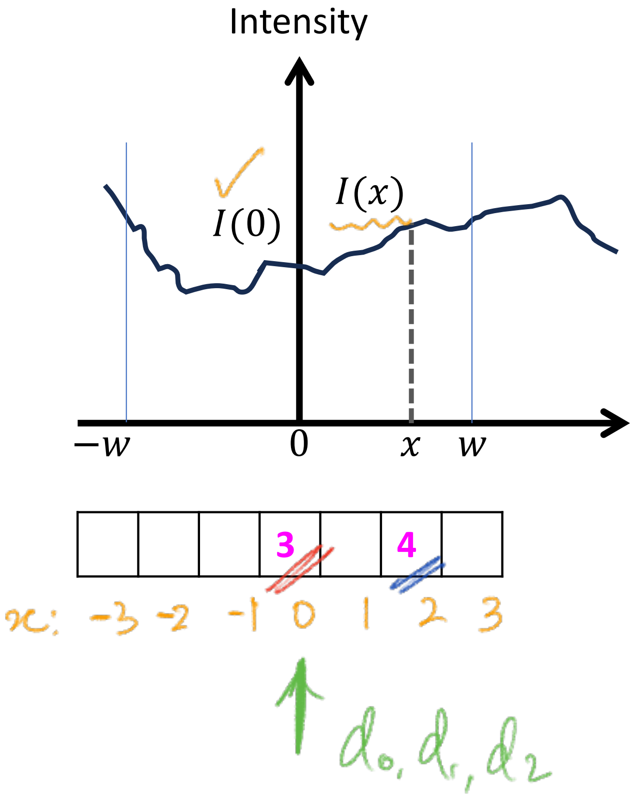
Fit a polynomial of degree n to the patch intensities

$f'(0), f''(0), \ldots$?



Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = \underline{I(0)} + \underline{xI'(0)} + \underline{\frac{1}{2}x^2 I''(0)}$$

$$I(x) = \begin{bmatrix}1 & x & \frac{x^2}{2}\end{bmatrix}\begin{bmatrix}d_0\\d_1\\d_2\end{bmatrix}$$

$X: 0, \ I(0) = 3$

$$\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1 & 2 & 2\end{bmatrix}\begin{bmatrix}d_0\\d_1\\d_2\end{bmatrix}$$

← 2 eq.

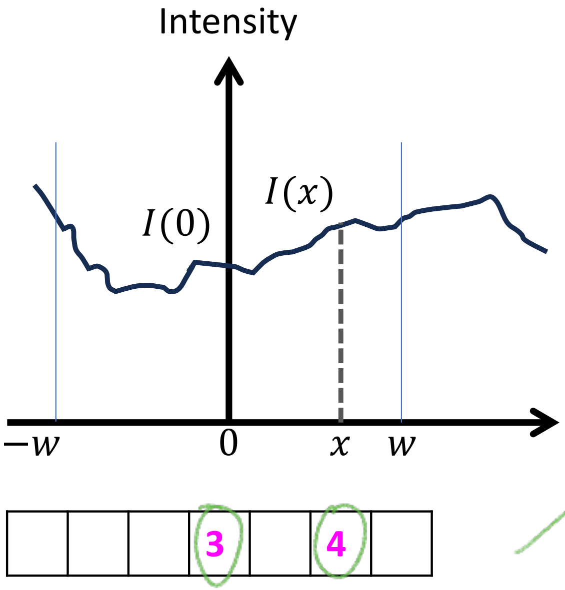
↑

3 unknowns

$X: 2, \ I(2) = 4$

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0)$$

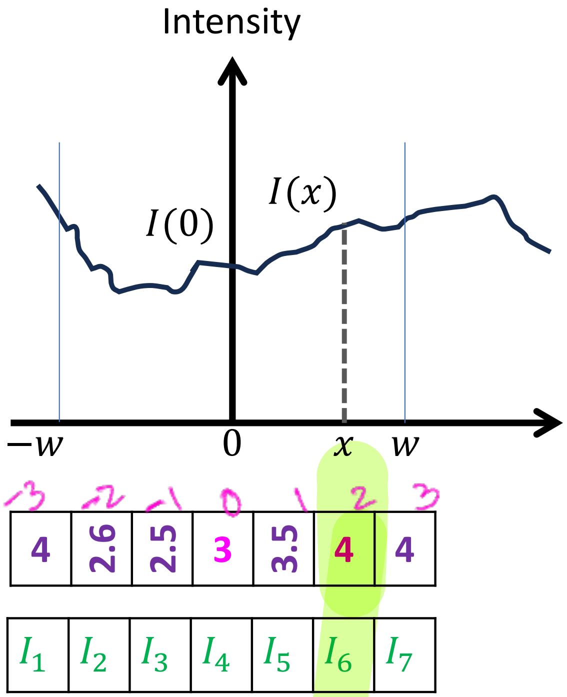
Unknowns

Using 2-pixels give me 2 equations.

With 7-pixels, I will get 7 equations.

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



For convenience, we refer to patch intensities as I_x where $x \in [1,2w+1]$. Then I_{w+1} refers to the intensity at patch center.

Fitting a polynomial of degree 2

Use second-order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0)$$

$$I(x) = \begin{bmatrix} 1 & x & \frac{x^2}{2} \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

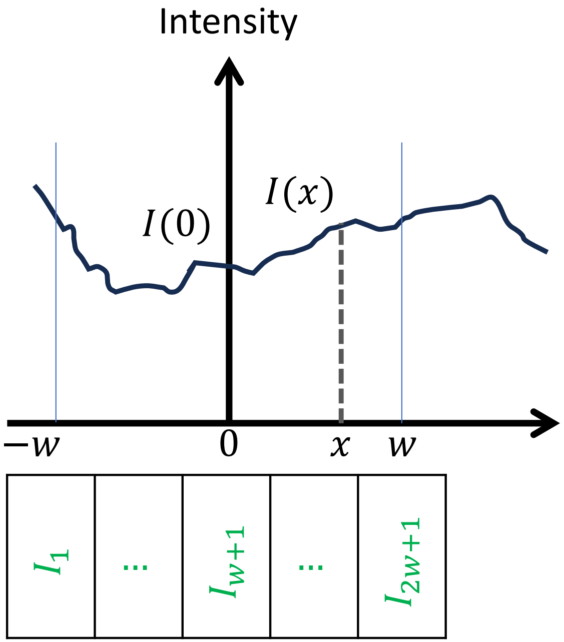
MATRIX FORM

$$\begin{array}{cccc} \vec{b} & & A & \vec{x} \\ \begin{bmatrix} 4 \\ 2.6 \\ 2.5 \\ 3 \\ 3.5 \\ 4 \\ 4 \end{bmatrix} & = & \begin{bmatrix} 1 & -3 & 9/2 \\ 1 & -2 & 2 \\ 1 & -1 & 1/2 \\ 1 & 0 & 0 \\ 1 & +1 & 1/2 \\ 1 & +2 & 2 \\ 1 & +3 & 9/2 \end{bmatrix} & \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} \end{array}$$

$$A\vec{x} = \vec{b}$$

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree n

Use nth order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2I''(0) + \frac{1}{6}x^3I'''(0) + \cdots + \frac{1}{n!}x^nI^{(n)}(0)$$

$(n+1)$ Unknowns

Observation

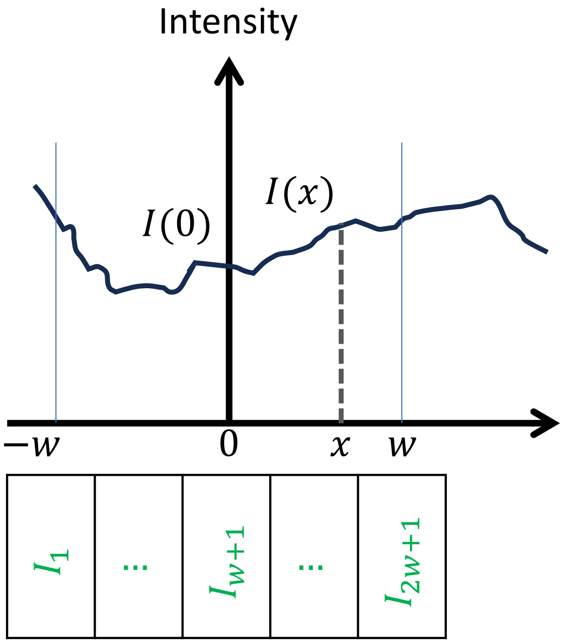
A $(2w+1)$-patch gives $2w+1$ equations.

Conclusion

For a patch of size $(2w+1)$, it is only possible to fit a polynomial of degree $2w$.

Compute derivatives at pixel 0 (i.e., the center of the patch)

Fit a polynomial of degree n to the patch intensities



Fitting a polynomial of degree n

Use nth order Taylor series expansion

$$I(x) = I(0) + xI'(0) + \frac{1}{2}x^2 I''(0) + \frac{1}{6}x^3 I'''(0) + \cdots + \frac{1}{n!}x^n I^{(n)}(0)$$

$(n+1)$ Unknowns

$$\boldsymbol{I}_{(2w+1)\times 1} = \boldsymbol{X}_{(2w+1)\times n}\boldsymbol{d}_{n\times 1}$$

Intensities (known)

Positions (known)

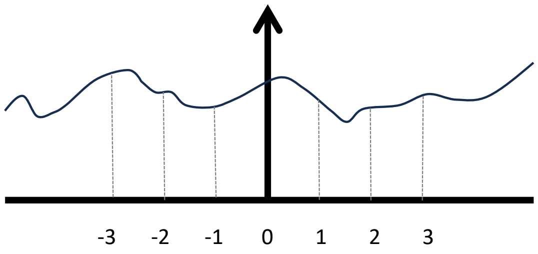
Derivatives (unknown)

Solve this linear system of equations in terms of $\boldsymbol{d}$ minimizes the fit error.

$$\boxed{\|\boldsymbol{I} - \boldsymbol{X}\boldsymbol{d}\|^2}$$

Solution $\boldsymbol{d}$ is called the *least squares fit*

0th order estimation (constant) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [d_0]$$

$I_5 = d_0$

Patch size = 7.

$$E = \sum_{i=1}^{7} (I_i - d_0)^2$$

$$= (I_1 - d_0)^2 + (I_2 - d_0)^2 + \cdots + (I_7 - d_0)^2$$

$\frac{dE}{d d_0} = 0$ solve for d_0

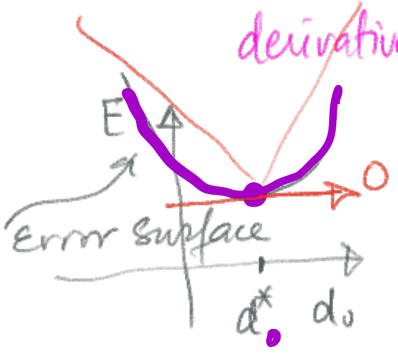
location

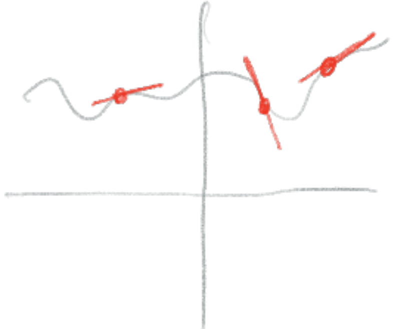
↓

$I(x) = I(0) = [1][d_0]$

↑

derivative





$$E = \sum_{i=1}^{7} (I_i - d_0)^2$$

let denot d_0 with x.

$$E = \underbrace{\sum_{i=1}^{7} (I_i - x)^2}$$

$$\frac{dE}{dx} = \sum_{i=1}^{7} 2(I_i - x)\underline{\underline{(-1)}}$$

$$\frac{dx^2}{dx} = 2x$$

$$\frac{d(-x)}{dx} = -1$$

$$\frac{dx}{dx} = 1$$

$$\left(\frac{d(I_i - x)}{dx} = -1 \right.$$

$$\Rightarrow \frac{dI_i}{dx} - \frac{dx}{dx} = -1$$

$$\frac{df(g(x))}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

set it equal to 0 and solve for x

$$\sum_{i=1}^{7} (-2)(I_i - x) = 0$$

$$\Rightarrow \underset{\neq 0}{\underline{\underline{-2}}} \sum_{i=1}^{7} (I_i - x) = 0$$

$$-2\, a = 0$$

$$\therefore \sum_{i=1}^{7} (I_i - x) = 0$$

$$\Rightarrow \sum_{i=1}^{7} I_i - 7x = 0$$

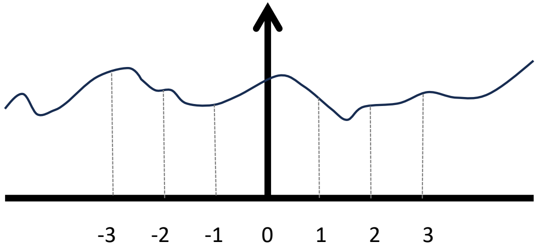
$$\Rightarrow 7x = \sum_{i=1}^{7} I_i$$

$$\Rightarrow x = \frac{1}{7} \sum_{i=1}^{7} I_i$$

$$\Rightarrow d_0^* = \frac{1}{7} \sum_{i=1}^{7} I_i \nearrow$$

ESTIMATE

0th order estimation (constant) of $I(x)$



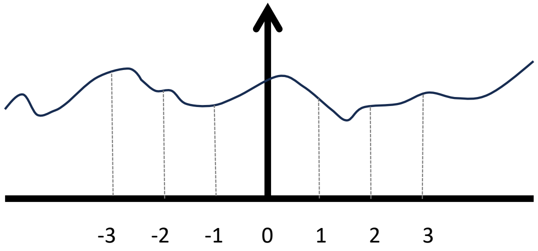
System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [d_0]$$

Solution is the mean intensity of the patch

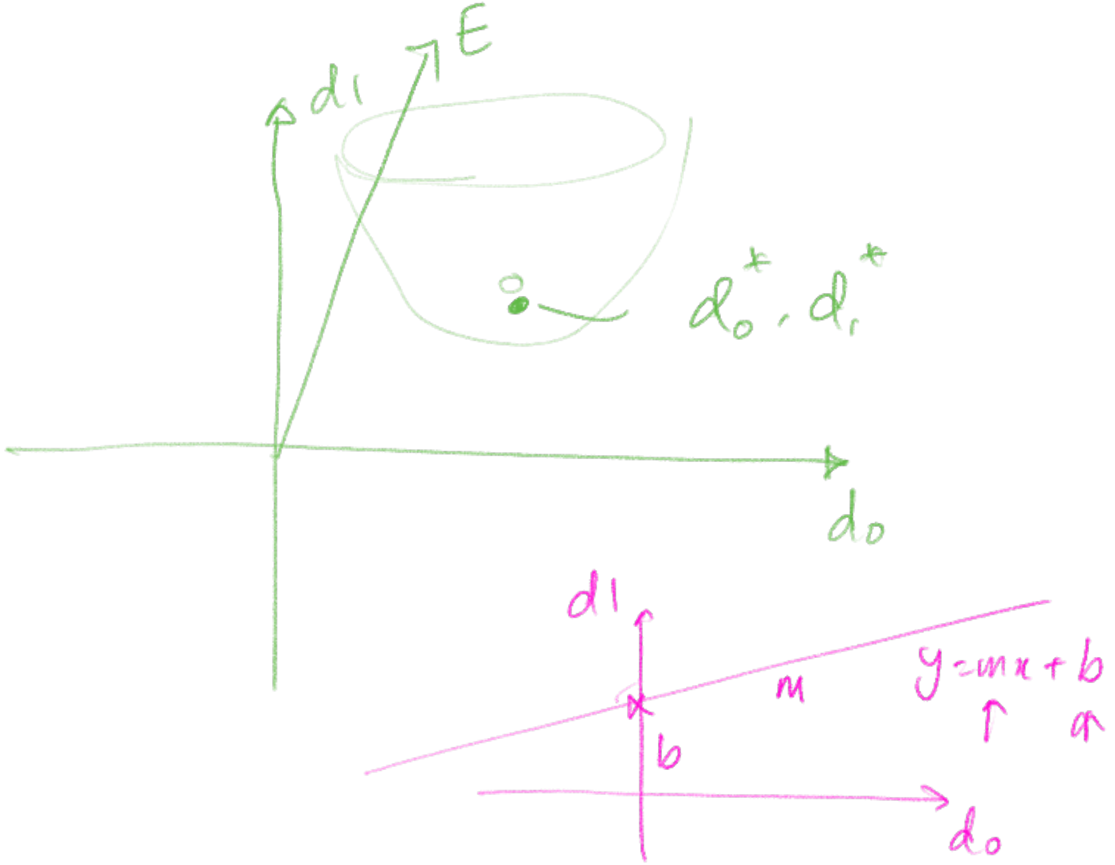
Provides the estimate of intensity of the center of the patch

1st order estimation (linear) of $I(x)$



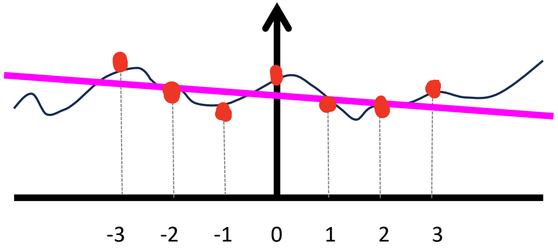
System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} \begin{array}{l} \leftarrow b \\ \leftarrow m \end{array}$$



1st order estimation (linear) of $I(x)$

$$E = \sum_{i=1}^{7} \left(I_i - (d_0 + x d_1) \right)^2$$



Solution minimizes the sum of vertical distance between the line and the image intensities.

Provides the estimate of intensity and its derivative at the patch center

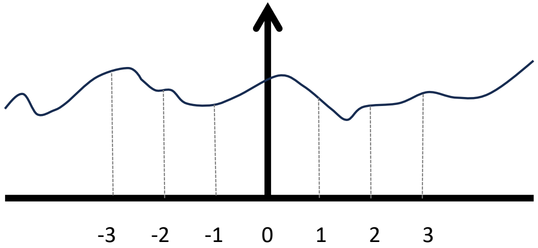
System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$

Matrix representation of a line (in 2D)

$$y = b + mx = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix}$$

2nd order estimation (quadratic) of $I(x)$

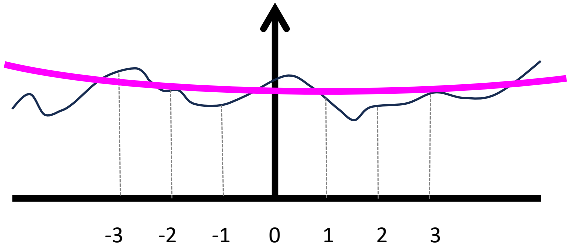


$$I(x) = I(0) + x I'(0) + x^2 I''(0)/2$$

System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & {}^{9}/_{2} \\ 1 & -2 & 2 \\ 1 & -1 & {}^{1}/_{2} \\ 1 & 0 & 0 \\ 1 & 1 & {}^{1}/_{2} \\ 1 & 2 & 2 \\ 1 & 3 & {}^{9}/_{2} \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

2nd order estimation (quadratic) of $I(x)$



System of linear equations that needs solving:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & {}^{9}/_{2} \\ 1 & -2 & 2 \\ 1 & -1 & {}^{1}/_{2} \\ 1 & 0 & 0 \\ 1 & 1 & {}^{1}/_{2} \\ 1 & 2 & 2 \\ 1 & 3 & {}^{9}/_{2} \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

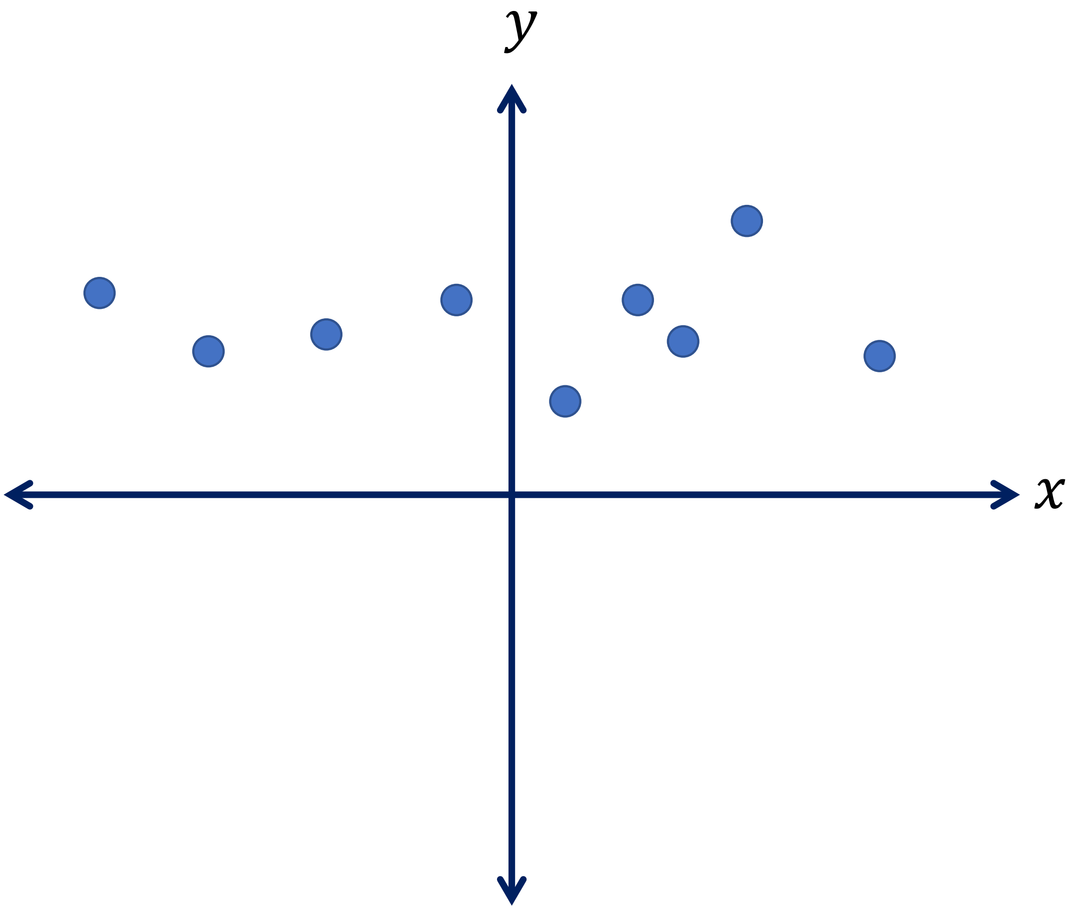
Solution fits a parabola/hyperbola/ellipse to patch intensities

Provides the estimate of intensity and its first and second derivatives at the patch center

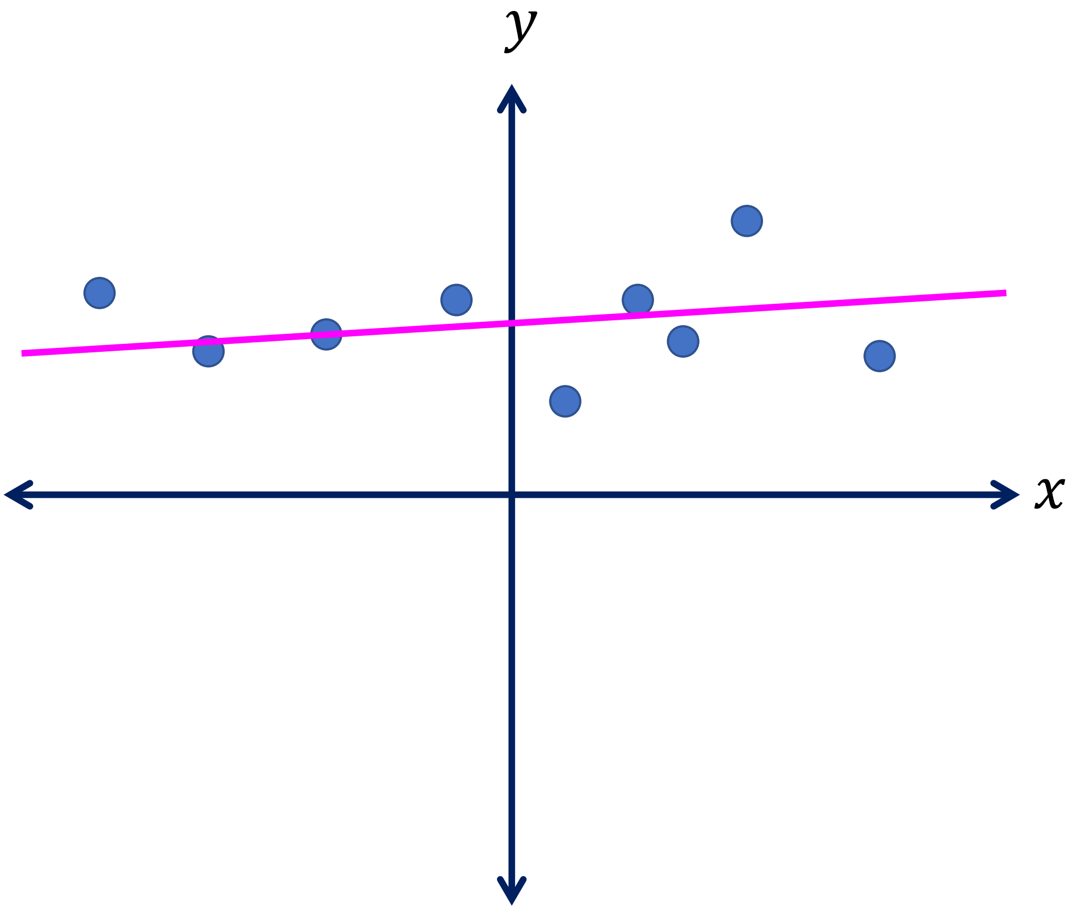
Matrix representation of second order polynomials

$$y = ax^2 + bx + c = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Least squares fitting



Least squares fitting



Least squares fitting often use the following notation to represent the system of linear equations

$$\boldsymbol{Ax} = \boldsymbol{b}$$

The solution is

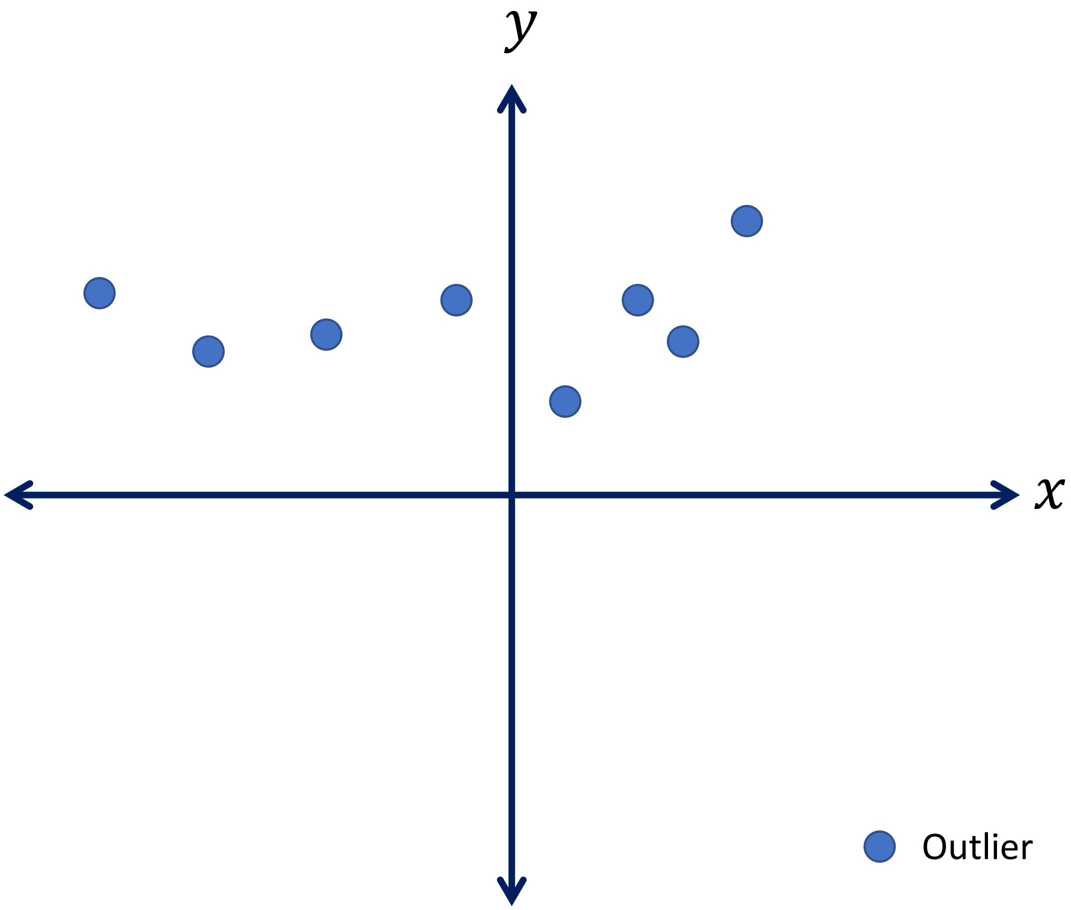
$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

where $\boldsymbol{A}^{-1}$ is inverse (or pseudo-inverse) of $\boldsymbol{A}$.

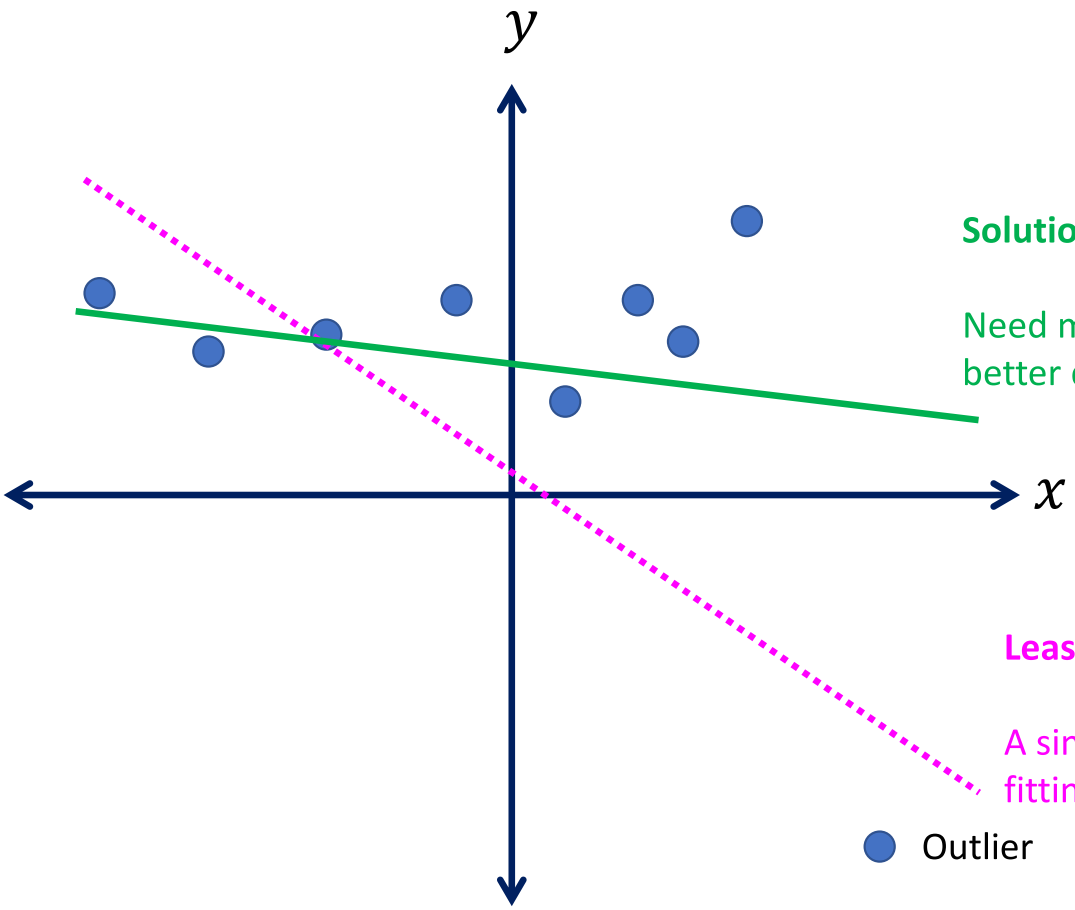
Recall that we need to solve the following system of linear equations when approximating patches with polynomials.

$$\underbrace{\boldsymbol{I}_{(2w+1)\times 1}}_{\boldsymbol{b}} = \underbrace{\boldsymbol{X}_{(2w+1)\times n}}_{\boldsymbol{A}}\underbrace{\boldsymbol{d}_{n\times 1}}_{\boldsymbol{x}}$$

Least squares fitting



Least squares fitting



Solution: Robust least squares fitting

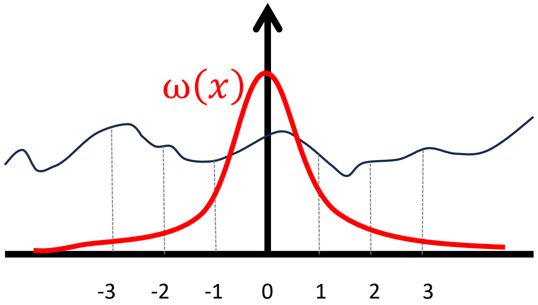
Need more completed methods that can better deal with outliers

Least squares fitting

A single outlier has resulted in a poor line-fitting estimate

● Outlier

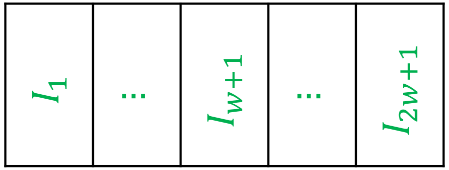
Weighted least squares estimate of $I(x)$



Give more weight to the pixels near center and less weight to pixels that are far from center, e.g., $\omega(x) = e^{-x^2}$

Bias our estimate of $I'(0)$ towards the center of the patch.

For patch



The system of linear equations becomes

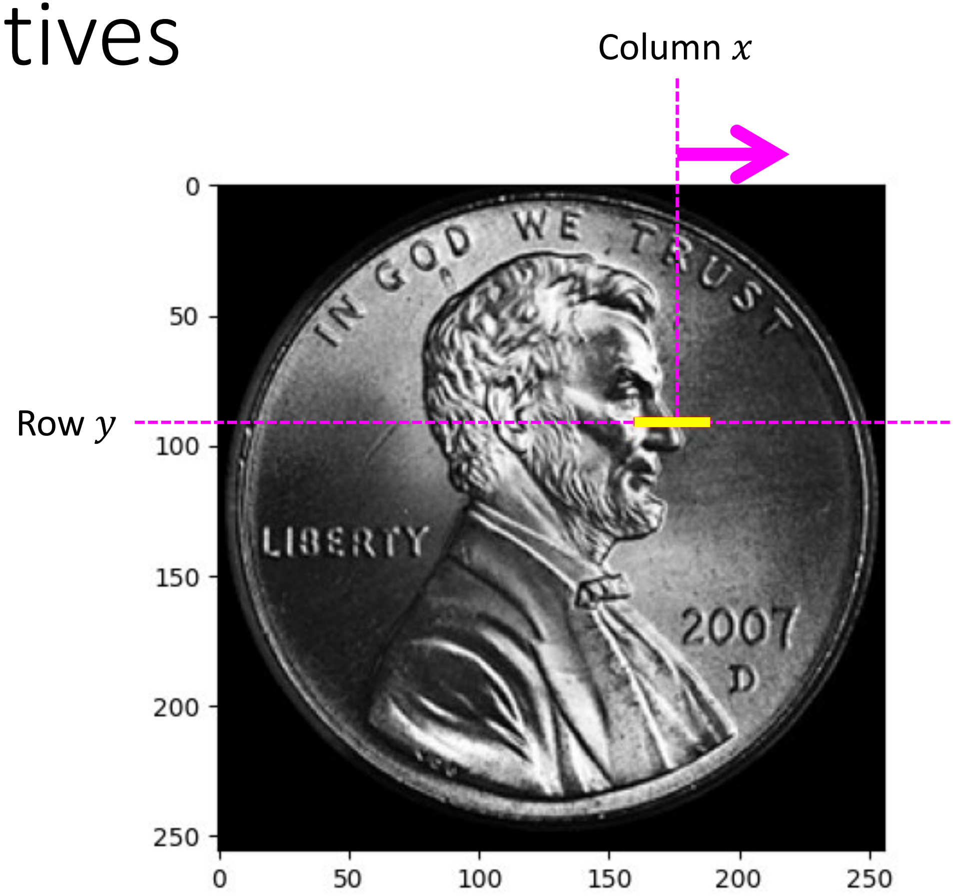
$$\begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{2w+1} \end{bmatrix} \boldsymbol{I}_{(2w+1)\times 1} = \begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{2w+1} \end{bmatrix} \boldsymbol{X}_{(2w+1)\times n} \boldsymbol{d}_{n\times 1}$$

and the solution $\boldsymbol{d}$ minimizes the norm:

$$\left\| \begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{2w+1} \end{bmatrix} (\boldsymbol{I} - \boldsymbol{X}\boldsymbol{d}) \right\|^2$$

Estimating image derivatives

- For each row y, define a window of width $2w + 1$ at pixel (i.e., column) x
  - Fit a polynomial (usually of degree 1 or 2)
  - Assign the fitted polynomial's derivates at location 0 (i.e., center of the patch, or column y in the image space)
  - Slide the window one over, until the end of the row



Summary

- 1D image patches
- Approximating 1D image patches via polynomials
- Computing image derivatives via fitting polynomials
- Least squares solution to a system of linear equations
- Weighted least squares