2D Curves in Images

Why do we care:

- object boundaries
- isophote: group of neighbouring pixels that have the same value.

Goal: mathematical representation of 2D curves.

* Parameterized 2D curves

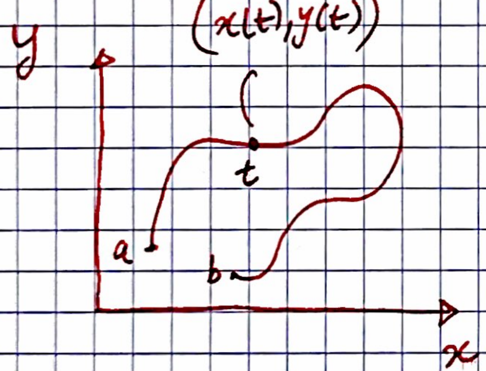
A continuous mapping $\gamma : (a, b) \rightarrow \mathbb{R}^2$

$t \rightarrow \underbrace{(x(t), y(t))}_{\text{location along the curve at position } t}$

value of parameter at the start

value of parameter at the end

parameter that indicates the position along the curve



Example: a boundary curve.

$t \rightarrow$ a pixel # along the boundary.

$x(t) \rightarrow$ x-coordinate of the t^{th} pixel.

$y(t) \rightarrow$ y-coordinate of the t^{th} pixel.

Observation: To fully describe a curve in 2D, we need two functions $x(t)$, $y(t)$. These are called the <u>coordinate functions</u>.

* Definition: A curve is called <u>smooth</u> when all derivatives $\frac{d^n x(t)}{dt^n}$ and $\frac{d^n y(t)}{dt^n}$ exist for all n and t.

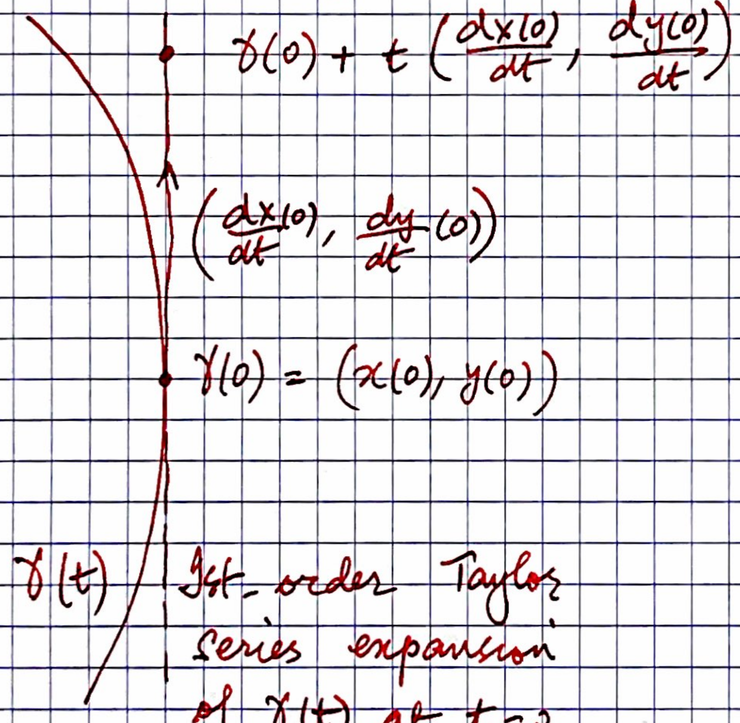
* Definition: A continuous function

$$\gamma : (a,b) \rightarrow \mathbb{R}^2$$
$$t \rightarrow (x(t), y(t))$$

is called a <u>closed curve</u> if $(x(a), y(a)) = (x(b), y(b))$

* The 1st and 2nd derivatives of $x(t), y(t)$ are extremely informative about curve shapes.

<u>Curve Geometry</u>



$$\gamma(t) = (x(t), y(t)) \cancel{\approx (x(0), y(0)) + t\left(\frac{dx(0)}{dt}, \frac{dy(0)}{dt}\right)}$$

$$\approx \left(x(0) + t\frac{dx(0)}{dt},\ y(0) + t\frac{dy(0)}{dt}\right)$$

$$= \underbrace{(x(0), y(0))}_{\text{point}} + t\underbrace{\left(\frac{dx(0)}{dt}, \frac{dy(0)}{dt}\right)}_{\text{tangent vector}}$$

Definition: Tangent vector at $\gamma(t)$ is

$$\frac{d\gamma(t)}{dt} = \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t)\right)$$

* Aside: Vector-valued function.

$\gamma(t)$ maps a number to a 2D point $(x(t), y(t))$

$\gamma(t)$ is therefore a vector-valued function.

Say we have a vector-valued function

$$f(t) = (f_1(t), f_2(t), \cdots, f_n(t))$$

then its derivative is

$$\frac{df(t)}{dt} = \left(\frac{df_1(t)}{dt}, \frac{df_2(t)}{dt}, \cdots, \frac{df_n(t)}{dt}\right)$$

* Effect of curve parameter t on tangent.

** Regardless of the parameter, the direction of the tangent remains unchanged.

Ex. possible parameter choices for a curve.

t = # pixels between $\gamma(0)$ and $\gamma(t)$.

or t = arc length of the curve between $\gamma(0)$ and $\gamma(t)$

Proof: We can parameterize a curve γ many different ways. Say we choose t = # pixels between $\gamma(0)$ and $\gamma(t)$ and $s = f(t)$. f is any differentiable function.

Then $$\frac{d\gamma}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

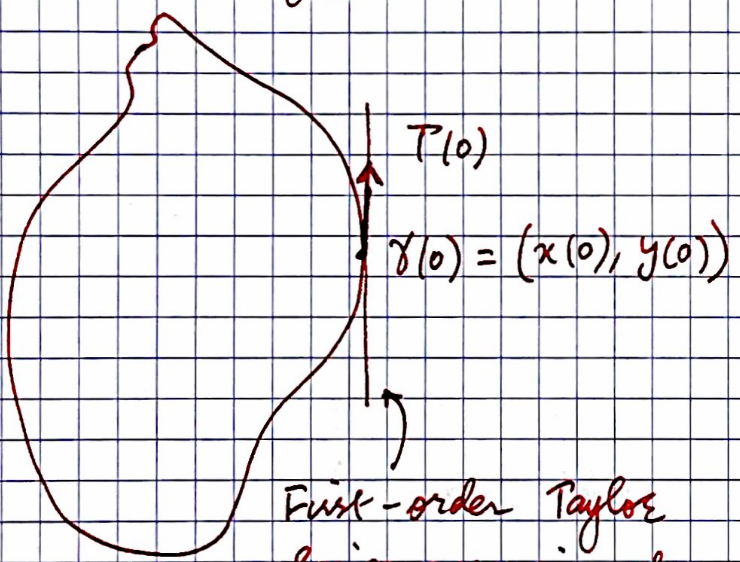
$\because s = f(t)$

$$\therefore \frac{d\gamma}{dt} = \left(\frac{dx}{ds}\frac{df}{dt}, \frac{dy}{ds}\frac{df}{dt}\right)$$

$$\frac{d\gamma}{dt} = \underbrace{\frac{df}{dt}}_{\text{multiplicative scalar}} \underbrace{\left(\frac{dx}{ds}, \frac{dy}{ds}\right)}_{\frac{d\gamma}{ds}}$$

Therefore, vectors $\frac{d\gamma}{dt}$ and $\frac{d\gamma}{ds}$ differ by a multiplicative factor only. The vectors are colinear.

The Unit Tangent Vector



First-order Taylor Series expansion of $\gamma(t)$ at $t=0$.

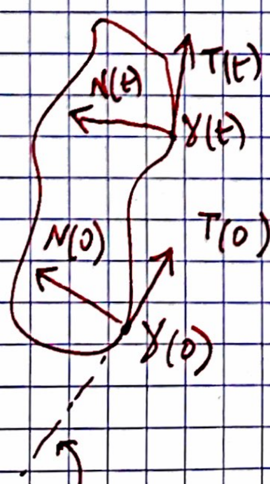
$$T(t) = \frac{d\gamma(t)}{dt} \cdot \frac{1}{\left\| \frac{d\gamma(t)}{dt} \right\|_2}$$

Recall that

$$\frac{d\gamma(t)}{dt} = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right)$$

The length of $T(0)$ is always 1.

The Unit Normal



- The unit vector is orthogonal to $T(t)$
- Corresponds to $90°$ counter-clockwise rotation.

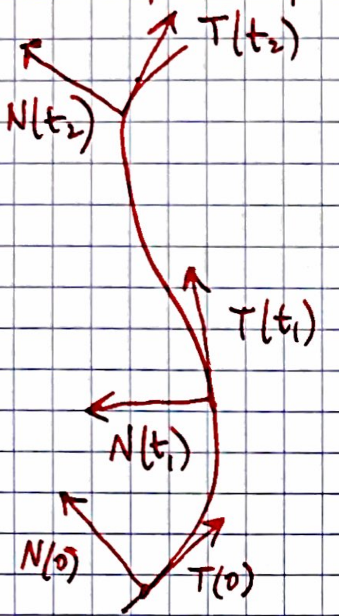
$$N(t) = \frac{1}{\left\| -\frac{dy(t)}{dt}, \frac{dx(t)}{dt} \right\|_2} \left(-\frac{dy(t)}{dt}, \frac{dx(t)}{dt}\right)$$

First order Taylor Series expansion of $\gamma(t)$ at $t=0$.

Definition: The Moving Frame

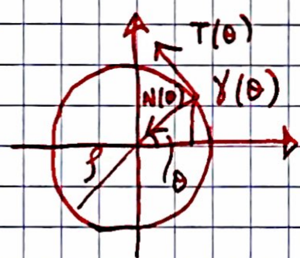
The pair of orthogonal vectors $\{T(t), N(t)\}$

Tangent ($T(t)$), Normal ($N(t)$)



As we change the parameter t, the moving frame rotates.
The *faster* the frame rotates, the more *curved* the curve is.

Example: Consider a circle.



$$\gamma(\theta) = \rho(\cos\theta, \sin\theta)$$
$$T(\theta) = (-\sin\theta, \cos\theta)$$
$$N(\theta) = (-\cos\theta, -\sin\theta)$$

\# Lets use Taylor series to analyze the moving frame.

$$\{T(t), N(t)\} = \{T(0), N(0)\} + \{\underbrace{t \cdot \frac{dT}{dt}(0)}_{(**)}, \underbrace{t \cdot \frac{dN}{dt}(0)}_{(***)}\}$$

Theorem: Definition of Curvature.

$$(**) \quad \frac{dT(t)}{dt} = k(t)\, N(t)$$

$$(***) \quad \frac{dN(t)}{dt} = -k(t)\, T(t)$$

$k(t)$ is a scalar, also called the curvature at t.

Definition of Curvature

$$\frac{dT}{dt} = k(t)\, N(t)$$

$$\frac{dN}{dt} = -k(t)\, T(t).$$

Proof: Let $T(t) = \big(u(t), v(t)\big)$ for some $u(t), v(t)$.

Then $\dfrac{dT(t)}{dt} = \left(\dfrac{du(t)}{dt}, \dfrac{dv(t)}{dt}\right)$

Length of $T(t) = 1$

Derivative of $(\text{length})^2 = 0$

$$\frac{d}{dt}\left(u^2(t) + v^2(t)\right) = 0$$

$$\Rightarrow \quad 2u(t) \cdot \frac{du(t)}{dt} + 2v(t) \cdot \frac{dv(t)}{dt} = 0$$

$$\Rightarrow \quad 2 \underbrace{\begin{bmatrix} \frac{du(t)}{dt} & \frac{dv(t)}{dt} \end{bmatrix}}_{\frac{dT(t)}{dt}} \underbrace{\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}}_{T(t)} = 0$$

$\dfrac{dT(t)}{dt}$ must be orthogonal to $T(t)$.

$\therefore \quad \dfrac{dT(t)}{dt}$ is collinear with $N(t)$.

$\therefore \quad \dfrac{dT(t)}{dt} \propto N(t)$

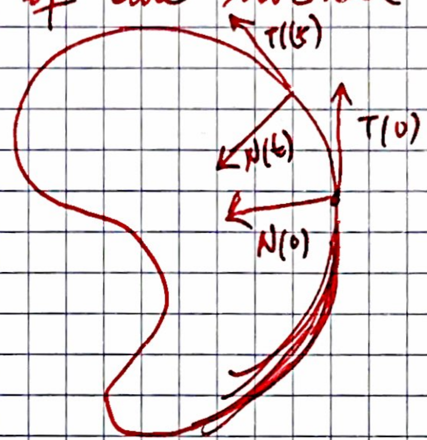
$$\Rightarrow \quad \frac{dT(t)}{dt} = k(t)\, N(t)$$

a scalar multiplier that happens to be the curvature.

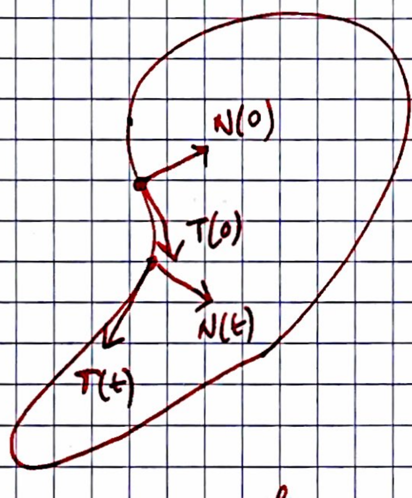
Curvature $k(t)$

$k(t) = 0$: straight line at t.

$k(t) > 0$: convex point, the curve bends in the direction of the normal.



$k(t) < 0$: concave point, the curve bends in the opposite direction from the normal.



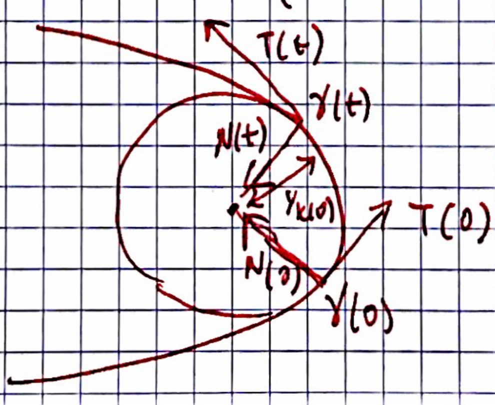
Convex curve is a curve whose points are all convex.

Higher values of $k(t)$ mean more curved the curve is.

* 1st order approximation of moving frame

$$\{T(t), N(t)\} = \{T(0), N(0)\} + \{t \cdot k(0) \cdot N(0),\ -t \cdot k(0) \cdot T(0)\}$$

(**)



Approximate $\gamma(t)$ by a circle that

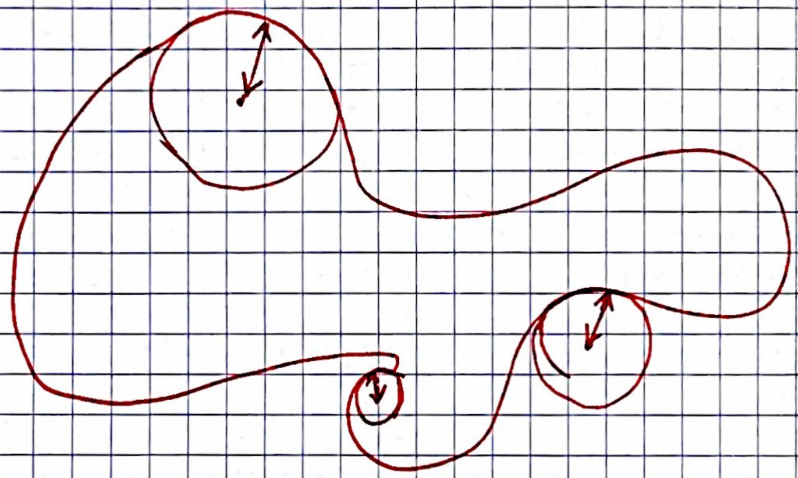
① passes through $\gamma(0)$

② passes through $\gamma(t)$

③ is tangent to $T(0)$

Then the radius of this circle is $1/\kappa(t)$.

This approximation implements (**) exactly.



<u>circle of curvature at $\gamma(t)$</u>

the circle that passes through $\gamma(t)$, is tangent to $T(t)$ and has radius $1/\kappa(t)$