Continuous Systems Simulations

- time → continuous variable
- Ordinary Differential Equations (ODE)
- Partial Differential Equations (PDE)

Example: Ball Throwing
- input: direction & force
- output: location
- state: location & velocity
- parameters: mass, gravity

ODEs
- Independent variable: derivatives w.r.t. this variable (time t)
- Dependent variables: functions of the independent variable. \( x = x(t) \)
- F that specifies the relationship between (A) + (B)
\[ F(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^n x}{dt^n}) \]
\[ \frac{dx}{dt}, x', Dx \rightarrow \text{denote derivatives.} \]

Order of an ODE = highest derivative in that eq.

Examples:
\[ mx'' = F \rightarrow \text{2nd order (Eq. of motion)} \]
\[ x' + 32x'' + x'' = 0 \]

*Key Idea to Solving ODEs* \( \rightarrow \) (Integration)

\[ x'' = 2 \]
\[ \downarrow \text{integrate once} \]
\[ x' = 2t + C_1 \]
\[ \downarrow \text{integrate again} \]
\[ x = t^2 + C_1t + C_2 \]
\[ \downarrow \text{Find out these values.} \]

**Initial Conditions**
\[ x(0) = 42 \]
\[ x'(0) = 7 \]

**Boundary Conditions**
\[ x'' = 2 \]
\[ x' = 2t + C_1 \]
\[ x = t^2 + C_1t + C_2 \]
\[
\begin{align*}
42 &= (0)^2 + C_1(0) + C_2 \\
\Rightarrow C_2 &= 42 \\
7 &= 2(0) + C_1 \\
\Rightarrow C_1 &= 7
\end{align*}
\]

**Boundary Conditions:**
\[
\begin{align*}
x(0) &= 42 \quad , \quad x(100) = 4300 \\
\Rightarrow C_2 &= 42 \\
4300 &= (100)^2 + C_1(100) + 42 \\
\Rightarrow C_1 &= \frac{4300 - (100)^2 - 42}{100}
\end{align*}
\]

**Reducibility**

**Case 1:** \( F(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}) \Rightarrow F(x, u, u\frac{du}{dt}) \)

where \( u = \frac{dx}{dt} \)

**Case 2:** \( F(t, \frac{dx}{dt}, \frac{d^2x}{dt^2}) \Rightarrow F(t, u, \frac{du}{dt}) \)

where \( u = \frac{dx}{dt} \)

A second-order equation is replaced by two first-order equations.
Stick to first order equations.

**Equation of Motion**

\[ a \propto F \quad \text{and} \quad a \propto \frac{1}{m} \]

\[ F = ma \]

\[ F = m \frac{dv}{dt} \quad \text{... Order 2} \]

Reduce

Introduce velocity \( v = \frac{dx}{dt} \) \quad \text{Order 1}

\[ F = m \frac{dv}{dt} \]

Solving \( F = ma \) numerically:

\[
\begin{bmatrix}
    v = \frac{\Delta x}{\Delta t} \\
    F = m \frac{\Delta v}{\Delta t}
\end{bmatrix}
\]

\[ \Rightarrow \begin{bmatrix}
    \Delta x = v \Delta t \\
    \Delta v = F \Delta t/m
\end{bmatrix} \]

What do we need: \( m = 1, \; F = 1 \)

\[ x(0) = 0, \; v(0) = 1, \; \Delta t = 1 \]

What is the value of \( x \) at time 2?
$\Delta v = (i)(1)/(1) = 1$

$v_{\text{new}} = v_{\text{old}} + \Delta v = 1 + 1 = 2$

$x_{\text{new}} = x_{\text{old}} + \Delta x = x_{\text{old}} + v_{\text{old}} \Delta t$

\[
= 0 + (1)(1)
= 1
\]