Rigid Body Dynamics

Simulation and Modeling (CSCI 3010U)

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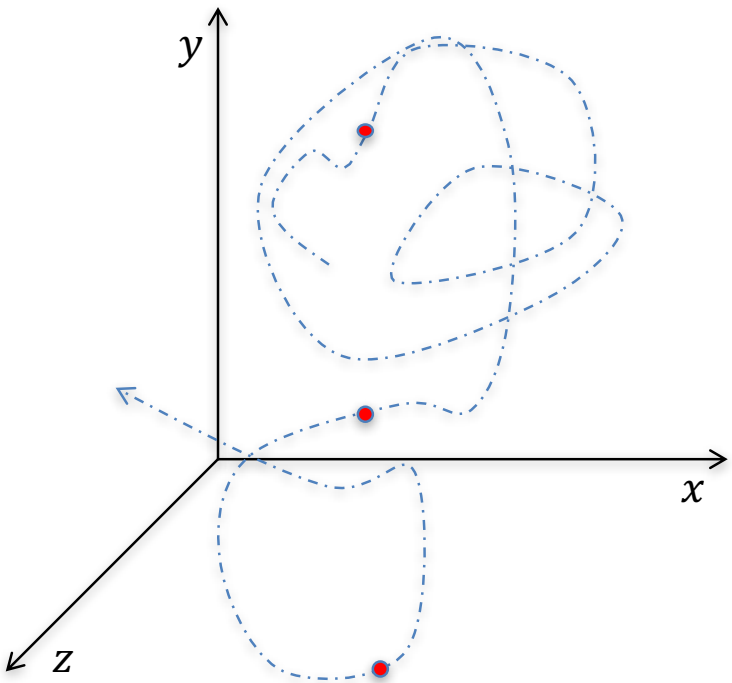
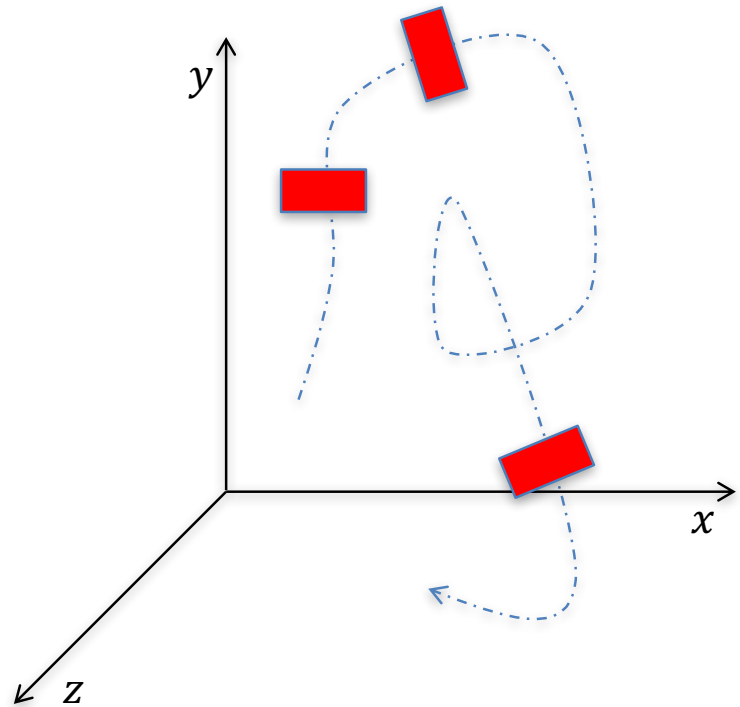
Rigid Bodies



Particle

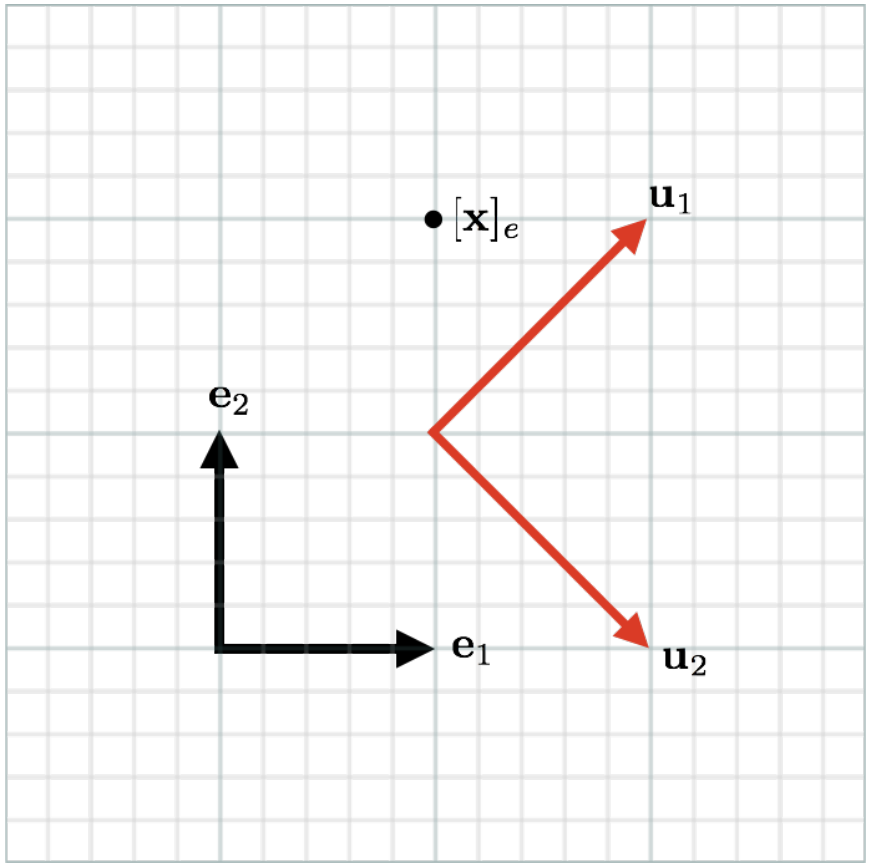


Rigid Body

Particle vs. Rigid Body Dynamics

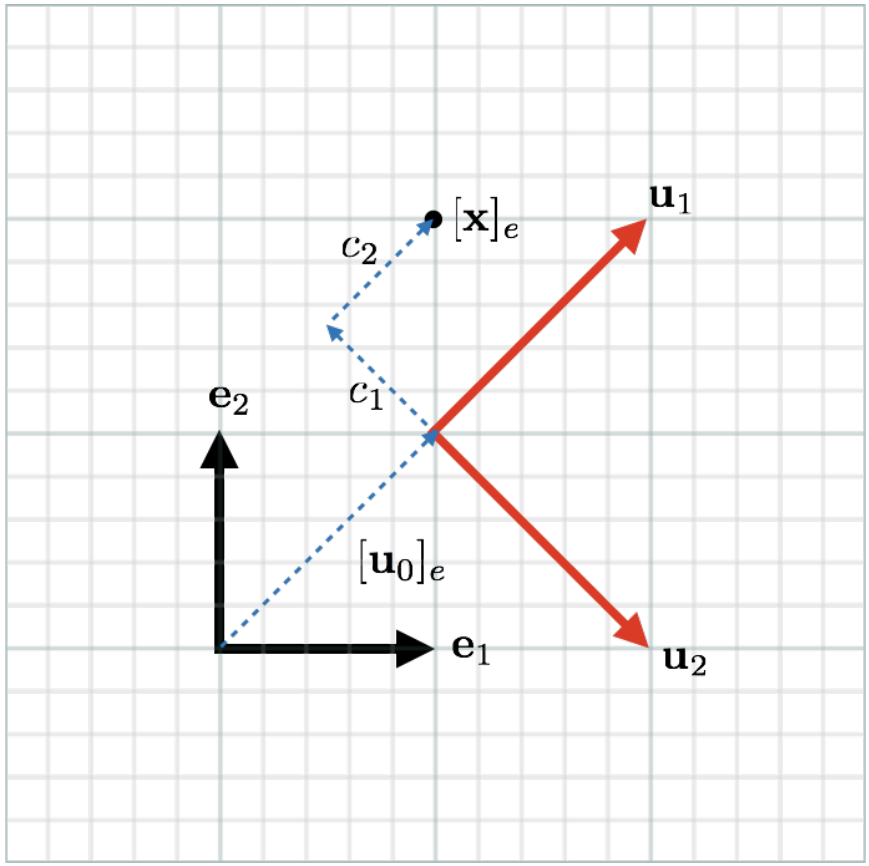
- State of a particle
  - Position $\mathbf{p}$
  - Velocity $\mathbf{v}$
- State of a rigid body
  - Position $\mathbf{p}$
  - Velocity $\mathbf{v}$
  - Orientation θ
  - Angular velocity ω

Coordinate frames



- $[\mathbf{x}]_e$ is $(1, 2)$ in coordinate frame described by $\mathbf{e}_1$ and $\mathbf{e}_2$
- What is $[\mathbf{x}]_u$, i.e., $[\mathbf{x}]_u$ expressed in $\mathbf{u}_1$ and $\mathbf{u}_2$?

Coordinate frames



- $\mathbf{x} = \mathbf{e}_1 + 2\mathbf{e}_2$
- $\mathbf{x} = [\mathbf{u}_0]_e + c_1[\mathbf{u}_1]_e + c_2[\mathbf{u}_2]_e$
- Note that $[\mathbf{x}]_u = (c_1, c_2)$, so we are interested in finding values of c_1 and c_2.

Coordinate frames

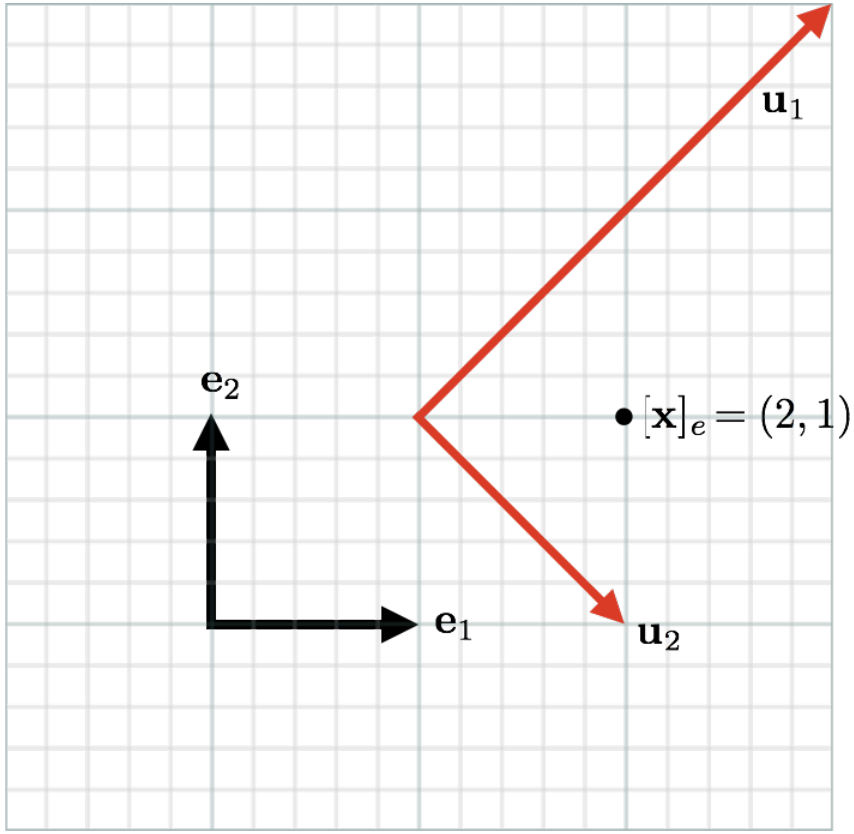
$$[\mathbf{u}_0]_e + c_1[\mathbf{u}_1]_e + c_2[\mathbf{u}_2]_e = [\mathbf{x}]_e$$
$$c_1[\mathbf{u}_1]_e + c_2[\mathbf{u}_2]_e = [\mathbf{x}]_e - [\mathbf{u}_0]_e$$
$$\begin{bmatrix} [\mathbf{u}_1]_e & [\mathbf{u}_2]_e \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} =$$
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} [\mathbf{u}_1]_e & [\mathbf{u}_2]_e \end{bmatrix}^{-1} ([\mathbf{x}]_e - [\mathbf{u}_0]_e)$$

Change of basis

$$[\mathbf{x}]_u = \begin{bmatrix} [\mathbf{u}_1]_e & [\mathbf{u}_2]_e \end{bmatrix}^{-1} ([\mathbf{x}]_e - [\mathbf{u}_0]_e)$$

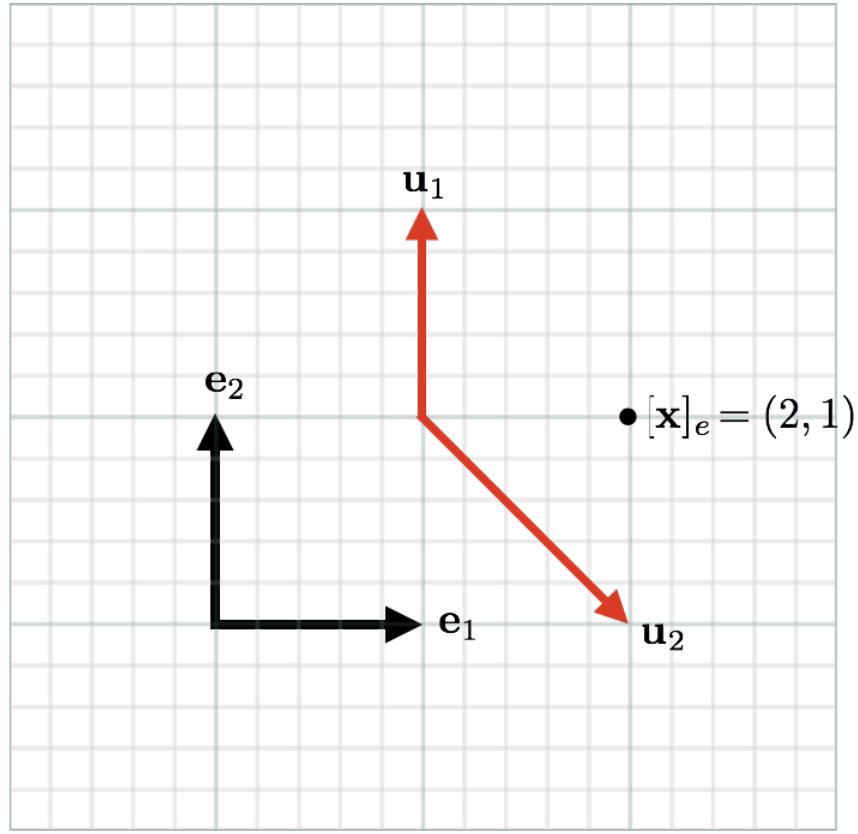
Coordinate Frames Exercise - Part 1

What is $[\mathbf{x}]_u$?

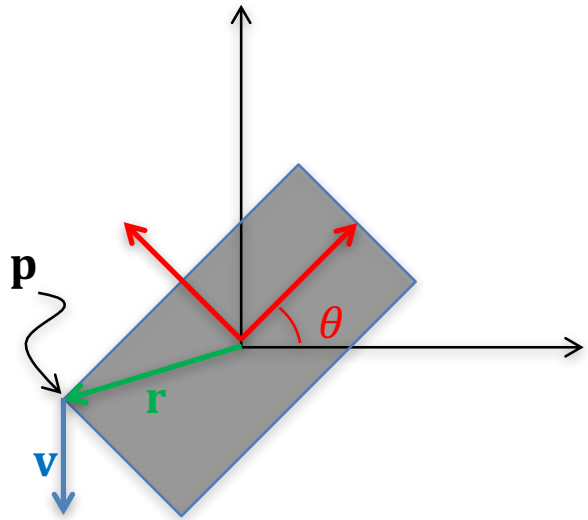


Coordinate Frames Exercise - Part 2

What is $[\mathbf{x}]_u$?



Rigid Bodies in 2D



Angular Velocity

- $\omega = \frac{d\theta}{dt}$
- Units are radians per second
- Radian is the angle subtended by an arc whose length is equal to its radious: $\theta = \frac{l}{r}$

Linear Velocity at a Point on the Body

- $\mathbf{v} = \mathbf{r}\omega$

Rigid Bodies in 3D

- Unlike 2D, orientation in 3D cannot be described using any angle.
- There are many schemes for describing rotations in 3D.
- We will use a 3x3 rotation matrix $\mathbf{R}$
- We need to find the relationship between angular velocity and rotation matrix.

We will return to this later.

Equations of Motions for Rigid Bodies

Force acting on a Rigid Body

- Net force acting on an object is the rate of change of its *linear momentum*.

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}$$

- Linear momentum: $\mathbf{P} = m\mathbf{v}$, where m is the mass of the object and $\mathbf{v}$ is its linear velocity

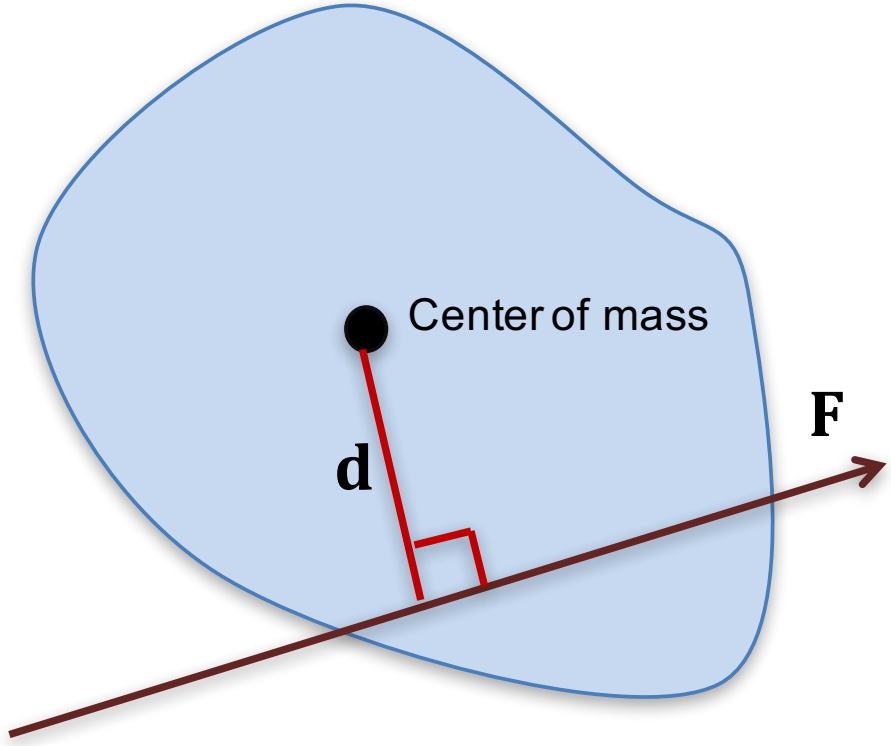
Torque acting on a rigid body

- Net torque acting on an object (about point $\mathbf{o}$) is the rate of change of its *angular momentum*.

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}$$

- Angular momentum: $\mathbf{L} = \mathbf{I}\omega$, where $\mathbf{I}$ is the *inertia tensor* and ω is its angular velocity (about point $\mathbf{o}$)

Torque

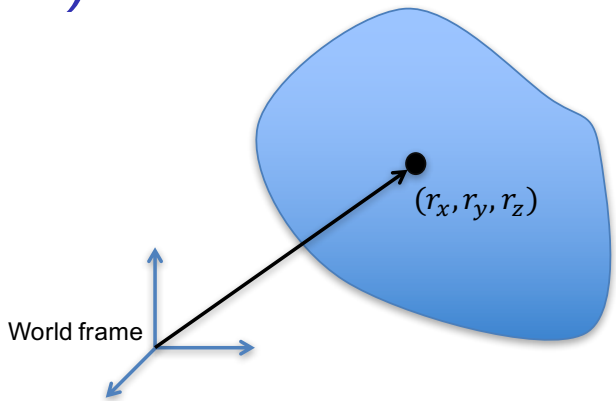


- Torque (in this example, clockwise or counter-clockwise):

$$\mathbf{T} = \mathbf{d} \times \mathbf{F}$$

- When force passes through the center of mass, the associated $\mathbf{d}$ vector is zero; therefore, this force produces no torque or rotational effect

Center of Mass (COM)



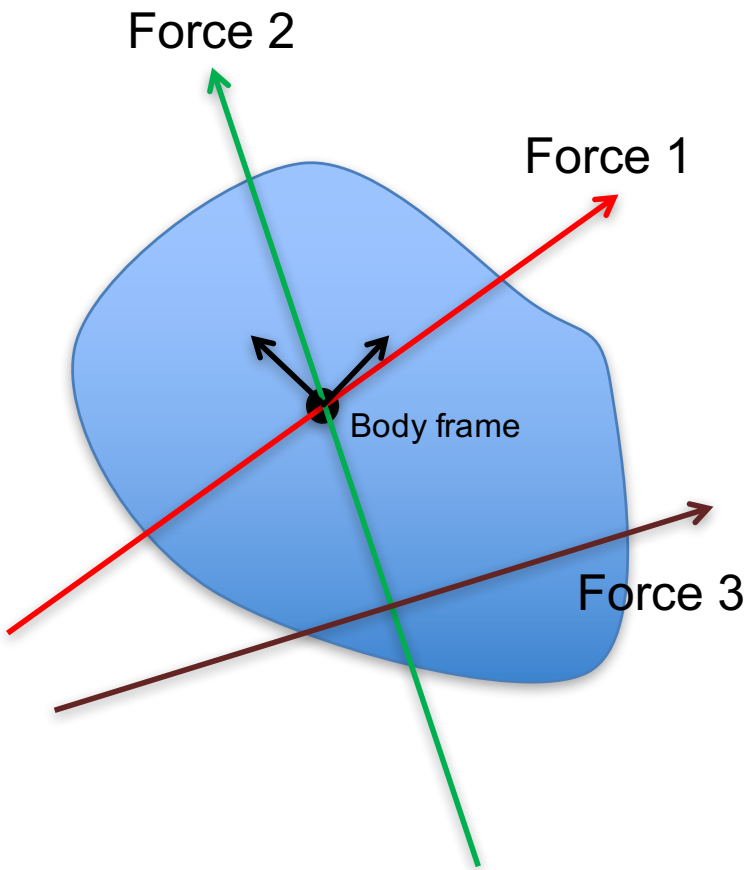
- The center of mass is the mean location of all the mass of the body.
- The center of mass $\mathbf{r}$ is defined as

$$r_x = \frac{1}{m} \int \rho(x, y, z) x dV$$
$$r_y = \frac{1}{m} \int \rho(x, y, z) y dV$$
$$r_z = \frac{1}{m} \int \rho(x, y, z) z dV$$

where $\rho(x, y, z)$ is the density at point (x, y, z).

COM as the origin of the body coordinate frame

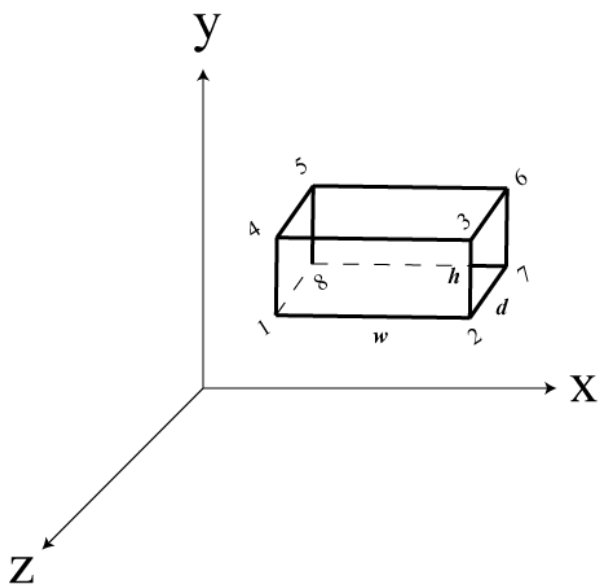
- Selecting COM as the origin of the body coordinate frame greatly simpifies the equation of motions
- Any force applied to (or passing through) the COM doesn't induce rotation.



- Force 1 results in translation only
- Force 2 results in translation only
- Force 3 results in both translation and rotation

COM - Discretization

Center of mass of a rectangular brick with point masses at its 8 vertices



- x_i, y_i and z_i are ith vertex location in the world coordinates.
- m_i is the value of the point mass at vertex i.

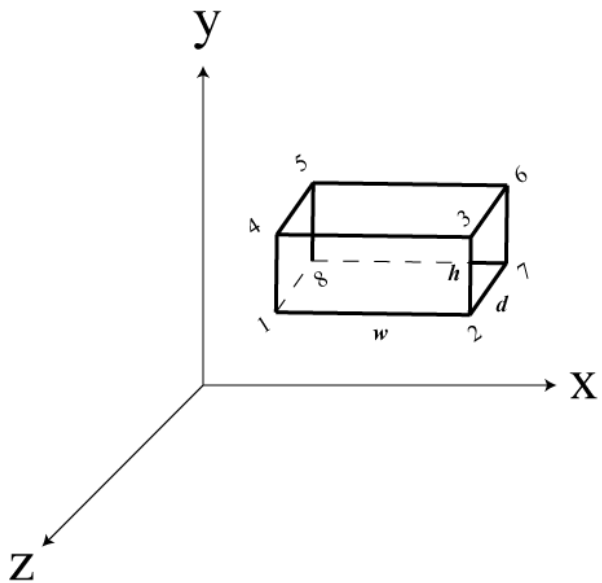
- (r_x, r_y, r_z) is the center of mass of the rectangular brick in the world coordiantes.

$$r_x = \left(\sum_i m_i x_i\right) / \left(\sum_i m_i\right)$$

$$r_y = \left(\sum_i m_i y_i\right) / \left(\sum_i m_i\right)$$

$$r_z = \left(\sum_i m_i z_i\right) / \left(\sum_i m_i\right)$$

COM - Example




```
% MATLAB Code
%    5 ----- 6
%   /       /|
%  4 ----- 3 7
%  |       |/
%  1 ----- 2
%
```

```
m = ones(1,8) / 8.;
```

```
r = zeros(8, 3);
r(1,:) = [1, 1, 1];
r(2,:) = r(1,:) + [w, 0, 0];
r(3,:) = r(2,:) + [0, h, 0];
r(4,:) = r(1,:) + [0, h, 0];
r(5,:) = r(4,:) + [0, 0, d];
r(6,:) = r(3,:) + [0, 0, d];
r(7,:) = r(2,:) + [0, 0, d];
r(8,:) = r(1,:) + [0, 0, d];
```

```
% compute center of mass first
M = sum(m);
```

```
com = ( m(1)*r(1,:) + ...
        m(2)*r(2,:) + ...
        m(3)*r(3,:) + ...
        m(4)*r(4,:) + ...
        m(5)*r(5,:) + ...
        m(6)*r(6,:) + ...
        m(7)*r(7,:) + ...
        m(8)*r(8,:) ) / M
```

Inertia Tensor

- Inertia tensor provides a concise description of the mass distribution around the center of mass

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

$$I_{xx} = \int \rho(x,y,z)(y^2 + z^2)dV$$

$$I_{yy} = \int \rho(x,y,z)(x^2 + z^2)dV$$

$$I_{zz} = \int \rho(x,y,z)(y^2 + x^2)dV$$

$$I_{xy} = \int \rho(x,y,z)xydV$$

$$I_{xz} = \int \rho(x,y,z)xzdV$$

$$I_{yz} = \int \rho(x,y,z)yzdV$$

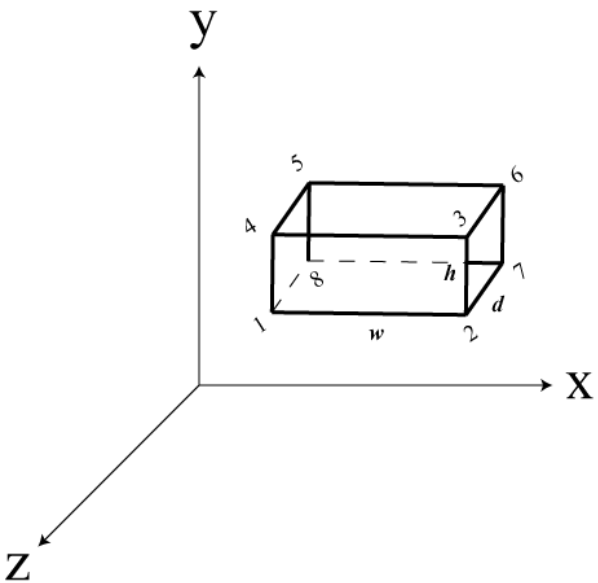
Inertia Tensor - Discretization

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

- $I_{xx} = \sum_i m_i(y_i^2 + z_i^2)$
- $I_{yy} = \sum_i m_i(z_i^2 + x_i^2)$
- $I_{zz} = \sum_i m_i(x_i^2 + y_i^2)$
- $I_{xy} = I_{yz} = \sum_i m_i x_i y_i$
- $I_{xz} = I_{zx} = \sum_i m_i x_i z_i$
- $I_{yz} = I_{zy} = \sum_i m_i y_i z_i$

- m_i are point masses.

Inertia Tensor - Example




```
% CONTINUED FROM PREVIOUS.
% com - center of mass (1x3)
% r - vertex locations (8x3)

% now lets compute inertia tensor
rp = r - repmat(com, 8, 1);

I = zeros(3,3);

mrp = repmat(m',1,3) .* rp

I(1,1) = rp(:,2)' * mrp(:,2) + rp(:,3)' * mrp(:,3);
I(2,1) = - rp(:,1)' * mrp(:,2);
I(3,1) = - rp(:,1)' * mrp(:,3);

I(1,2) = I(2,1);
I(2,2) = rp(:,1)' * mrp(:,1) + rp(:,3)' * mrp(:,3);
I(3,2) = - rp(:,2)' * mrp(:,3);

I(1,3) = I(3,1);
I(2,3) = I(3,2);
I(3,3) = rp(:,1)' * mrp(:,1) + rp(:,2)' * mrp(:,2);
```

Inertia Tensor in the Body Coordinate Frame

- The inertia tensor $\mathbf{I}$ that we just computed is expressed in the world coordinate frame. Consequently it changes as the orientation of the rigid body changes.
- We can express the inertia tensor in the body coordinate frame.
- We refer to inertia tensor in the body coordinate frame as $\mathbf{I}_{body}$.
- $\mathbf{I}_{body}$ doesn't change as the orientation of the body changes.
- $\mathbf{I}_{body}$ is diagonal, i.e.,

$$\mathbf{I}_{body} = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$$

- Inverse of $\mathbf{I}_{body}$:

$$\mathbf{I}_{body}^{-1} = \begin{bmatrix} \frac{1}{I_{11}} & 0 & 0 \\ 0 & \frac{1}{I_{22}} & 0 \\ 0 & 0 & \frac{1}{I_{33}} \end{bmatrix}$$

Computing $\mathbf{I}_{body}$

Option 1

Diagonalize $\mathbf{I}$

- Compute eigenvectors and eigenvalues of $\mathbf{I}$
- Eigenvalues form the diagonal matrix $\mathbf{I}_{body}$
- Eigenvectors form the 3-by-3 rotation matrix $\mathbf{R}$ that describes the orientation of the rigid body
- This is the preferred approach

Option 2

Use 3-by-3 rotation matrix $\mathbf{R}$ that describes the orientation of the rigid body

- $\mathbf{I}_{body} = \mathbf{R}^T\mathbf{I}\mathbf{R}$

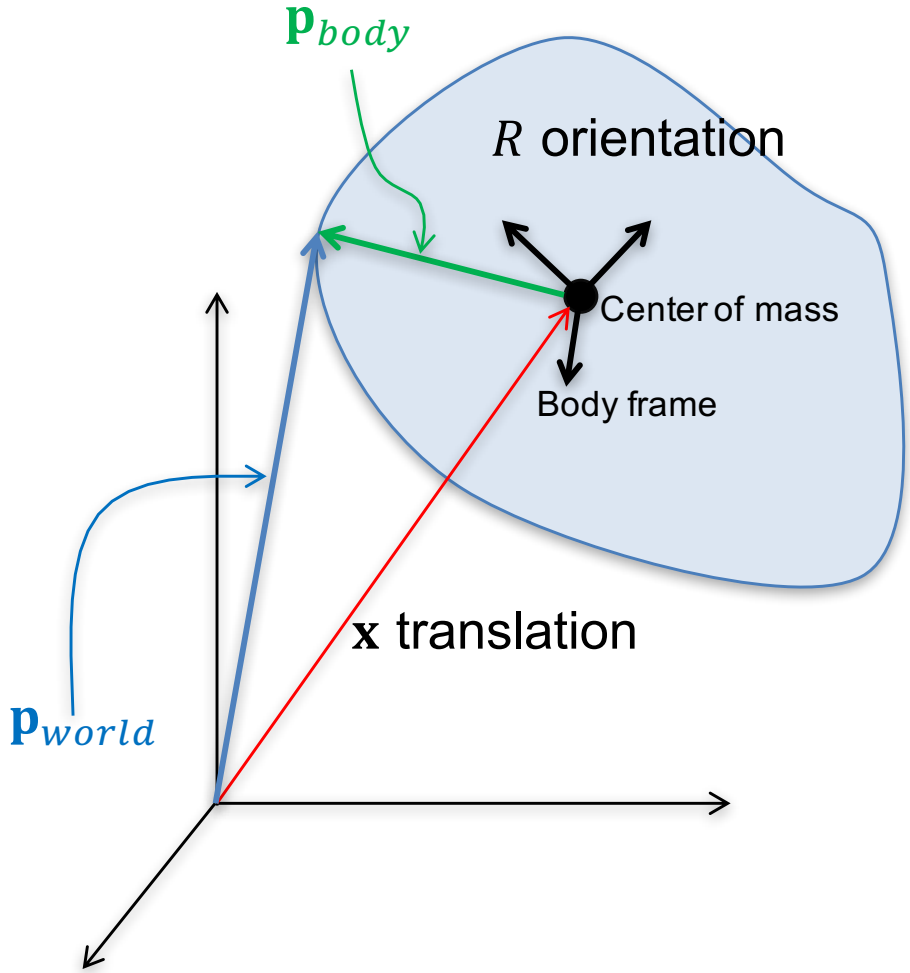
Inertia Tensor

- Inertia tensors are available for many canonical objects: rectangles, circles, spheres, etc.
- Efficient algorithms exist to compute inertia tensor, center of mass, body coordinate frames a given polygonal model of an object
- Many tools exist to construct polygonal models of 2D/3D rigid objects

Body coordinate frame

Attach a coordinate frame with each rigid body

- Origin = center of mass (defined in the world frame)
- Axes = defined in the world coordinate frame by a 3-by-3 rotation matrix $\mathbf{R}$. Columns of $\mathbf{R}$ define the x, y and z axes of the body coordinate frame
- Inertia tensor $\mathbf{I}_{body}$ is constant and diagonal in this frame



From body coordinate frame to world coordinate frame.

$$\mathbf{p}_{world} = \mathbf{R}\mathbf{p}_{body} + \mathbf{x}$$

World and Body Coordinate Frames

World coordinate frame

- Collision detection and response
- Display and visualization

Body coordinate frame

- Compute quantitites such as inertia tensor once and store them for later use.

Rigid Body Dynamics

State variables

Position	$\mathbf{x}$	1 by 3 vector
Orientation	R	3 by 3 rotation matrix
Linear Momentum	$\mathbf{P}$	1 by 3 vector
Angular Momentum	$\mathbf{L}$	1 by 3 vector

Constants

Mass	m	scalar
Inertia tensor	I_{body}	3 by 3 matrix (in body frame)

Derived quantities

Linear velocity	v	1 by 3 vector
Angular velocity	ω	1 by 3 vector
Inertia tensor	I^{-1}	3 by 3 matrix (in world frame)

Total force	$\mathbf{F}$	1 by 3 vector
Total torque	$\mathbf{T}$	1 by 3 vector

Rigid Body Dynamics

Linear effects

- $d\mathbf{x}/dt = \mathbf{v}$
- $d\mathbf{P}/dt = \mathbf{F}$
- $\mathbf{v} = \mathbf{P}/m$

Angular effects

- $d\mathbf{R}/dt = \omega^* \mathbf{R}$, where

$$\begin{bmatrix} 0 & -\omega_x & \omega_y \\ \omega_z & 0 & -\omega_x \\ \omega_y & \omega_x & 0 \end{bmatrix}$$

- $d\mathbf{L}/dt = \mathbf{N}$
- $\omega = \mathbf{I}^{-1}\mathbf{L}$
- $\mathbf{I}^{-1} = \mathbf{R}\mathbf{I}_{body}^{-1}\mathbf{R}^T$

Rigid Body Dynamics

```
// x - position
state[0] = x[0];
state[1] = x[1];
state[2] = x[2];

// R - orientation
state[3] = R[0][0];
state[4] = R[1][0];
state[5] = R[2][0];
state[6] = R[0][1];
state[7] = R[1][1];
state[8] = R[2][1];
state[9] = R[0][2];
state[10] = R[1][2];
state[11] = R[2][2];

// P - linear momentum
state[12] = P[0];
state[13] = P[1];
state[14] = P[2];

// L - angular momentum
state[15] = L[0];
state[16] = L[1];
state[17] = L[2];

// t – OSP needs it
state[18] = 0.0
```

Init: Flatten state variables into a state vector

```
// dx/dt = v
rate[0] = v[0];
rate[1] = v[1];
rate[2] = v[2];

// dR/dt = w* R
double[][] Rdot =
mult(star(omega), R);
rate[3] = Rdot[0][0];
rate[4] = Rdot[1][0];
rate[5] = Rdot[2][0];
rate[6] = Rdot[0][1];
rate[7] = Rdot[1][1];
rate[8] = Rdot[2][1];
rate[9] = Rdot[0][2];
rate[10] = Rdot[1][2];
rate[11] = Rdot[2][2];

// dP/dt = force
rate[12] = force[0];
rate[13] = force[1];
rate[14] = force[2];

// dL/dt = torque
rate[15] = torque[0];
rate[16] = torque[1];
rate[17] = torque[2];

// dt/dt = 1
rate[18] = 1;
```

Rate[] encodes 1st order ODE for our system

```
odeSolver.step();

// x
x[0] = state[0];
x[1] = state[1];
x[2] = state[2];

// R
R[0][0] = state[3];
R[1][0] = state[4];
R[2][0] = state[5];
R[0][1] = state[6];
R[1][1] = state[7];
R[2][1] = state[8];
R[0][2] = state[9];
R[1][2] = state[10];
R[2][2] = state[11];
R = orthonormalize(R);


// P
P[0] = state[12];
P[1] = state[13];
P[2] = state[14];

// L
L[0] = state[15];
L[1] = state[16];
L[2] = state[17];

Iinv = mult(R, mult(IbodyInv, transpose(R)));
omega = mult(Iinv, L);
```

Let ODE solve the state and then copy the state back to our state variables $\mathbf{x}$, R, $\mathbf{L}$ and $\mathbf{T}$.

Rigid Body Dynamics: Numerical Considerations

- Over time numerical errors accumulate in rotation matrix $\mathbf{R}$
- This effects our computation of $\mathbf{I}$ and ω
- Orthonormalize $\mathbf{R}$ after every timestep

Orthonormalization

1. Normalize $\mathbf{R}_1$ (excluding last column)
2. $\mathbf{R}_1 \times \mathbf{R}_2 = \mathbf{R}_3$ (normalize)
3. $\mathbf{R}_3 \times \mathbf{R}_1 = \mathbf{R}_2$ (normalize)

Here $\mathbf{R}_i$ represent the i-th row of matrix $\mathbf{R}$

Errors were shifted in the matrix

Representing Rotations

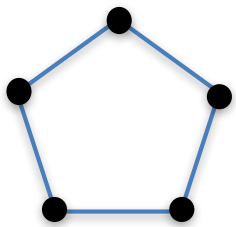
- We chose to represent rotations as 3-by-3 rotation matrices
- Quaternions can be used to represent rotations
- Most rigid body dynamics systems use quaterions
- See Ch. 17 of the textbook

Representing Rigid Bodies

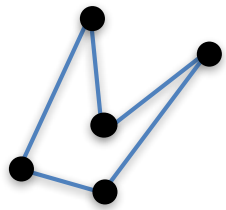
- A rigid body has a shape that does not change over time
- It can translate through space and rotate
- A rigid body occupies a volume of space
- The distribution of its mass over this volume determines its motion or dynamics
- Shape representation is studied extensively in computer graphics and some areas of mechanical engineering and mathematics
- There are many ways of representing shape, each with a different set of advantages and disadvantages
- We will stick to polygons

Shape Representation using Polygons

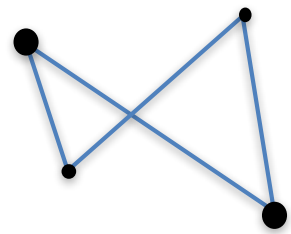
- The surface of the object is represented by a collection of polygons
- The polygons are connected across their edges to form a continuous surface
- In order to have a well behaved representation we need to constrain our polygons
- First all of the polygons must be convex, note that we can always convert a concave polygon into two or more convex ones



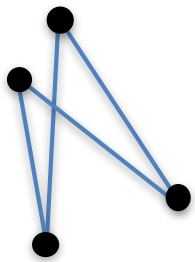
Convex Polygon



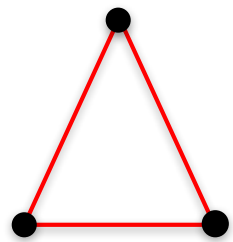
Concave Polygon



Non-planar Polygon



Self-intersecting Polygon



Triangle

Shape Representation using Triangles

- We will stick to triangles
- Any convex polygon can be converted to a collection of triangles

Advantages

- We are only dealing with one type of polygon, a uniform representation
- Triangles are the simplest polygon, makes our algorithms simpler
- Many modeling programs allow us to construct polygonal models
- Easy to display
- Many efficient algorithms exist for manipulating triangles

Disadvantages

- Not a compact representation
- Not a good approximation for curved surfaces

Other Types of Dynamics

- Articulated figures
  - Rigid bodies connected by joints and hinges
  - Used to model the dynamics of human figures
- Vehicle dynamics used to model the dynamics of various kinds of vehicles
- Deformable objects
  - Cloth, soft toys, etc.
- These are more complicated than what we have seen so far

Readings

- Ch. 17 of the textbook
- *Classical Mechanics (3rd Edition)* by H. Goldstein and C.P. Poole Jr.